# On deep learning based approximation algorithms for partial differential equations 

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> Thomas Müller-Gronbach (University of Passau, Germany),
> Diyora Saliomva (ETH Zurich, Switzerland), and
> Larisa Yaroslavtseva (University of Passau, Germany)
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November 30, 2017

Introduction

## Consider

$$
\begin{equation*}
\frac{\partial u}{\partial t}(t, x)+f\left(x, u(t, x),\left(\nabla_{x} u\right)(t, x),\left(\operatorname{Hess}_{x} u\right)(t, x)\right)=0 \tag{PDE}
\end{equation*}
$$

$$
\text { and } u(T, x)=g(x) \text { for } t \in[0, T), x \in \mathbb{R}^{d} \text { where } T>0, d \in \mathbb{N}
$$

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f: \mathbb{R}^{d} \times \mathbb{R} \times \mathbb{R}^{d} \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}, u \in C\left([0, T] \times \mathbb{R}^{d}, \mathbb{R}\right) \text { satisfies }
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$$
\left.u\right|_{[0, T) \times \mathbb{R}^{d}} \in C^{1,2}\left([0, T) \times \mathbb{R}^{d}, \mathbb{R}\right) . \text { Goal: Solve (PDE) approximatively. }
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## Applications: Pricing of financial derivatives,

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Linear PDEs

## Theorem (Hairer, Hutzenthaler, \& J 2015 AOP)

Let $T \in(0, \infty), d \in\{4,5, \ldots\}, \xi \in \mathbb{R}^{d}$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion W : $[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

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every $Y^{N}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{4}, N \in \mathbb{N}$, with $\forall N \in \mathbb{N}, n \in\{0,1, \ldots, N-1\}: Y_{0}^{N}=X_{0}$ and

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(Euler-Maruyama approximations), and every $\alpha \in[0, \infty)$ we have

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\lim _{N \rightarrow \infty}\left(N^{\alpha}\left\|\mathbb{E}\left[X_{T}\right]-\mathbb{E}\left[Y_{N}^{N}\right]\right\|\right)= \begin{cases}0 & : \alpha=0 \\ \infty & : \alpha>0\end{cases}
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Let $T \in(0, \infty), d \in\{4,5, \ldots\}, \xi \in \mathbb{R}^{d}$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion W: $[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

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Let $T \in(0, \infty), d \in\{2,3,4, \ldots\}, \xi \in \mathbb{R}^{d},\left(a_{N}\right)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy $\lim _{N \rightarrow \infty} a_{N}=0$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

$$
\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi,
$$

and every $N \in \mathbb{N}$ we have

- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
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## Theorem (Hefter \& J 2017)

Let $T, \delta, \beta \in(0, \infty), \gamma, \xi \in[0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W:[0, T] \times \Omega \rightarrow \mathbb{R}$ be a Brownian motion, let $X:[0, T] \times \Omega \rightarrow \mathbb{R}$ be a solution of

$$
\frac{\partial}{\partial t} x_{t}=\left(\delta-\gamma x_{t}\right)+\beta \sqrt{X_{t}} \frac{\partial}{\partial t} W_{t}, \quad i \in[0, T], \quad x_{0}=\xi
$$

Then there exists a $c \in(0, \infty)$ such that for all $N \in \mathbb{N}$ we have

$$
\left.\inf _{\substack{\text { unix } \\ \text { measurable }}} \mathbb{E}\left[\left\lvert\, X_{T}-u\left(W_{T}, W_{\frac{1}{N}}, \ldots, W_{T}\right)\right.\right]^{\top}\right] \geq c \cdot N \min ^{\left.-1, \frac{28}{P^{2}}\right\}} \text {. }
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$$

Then there exists a $c \in(0, \infty)$ such that for all $N \in \mathbb{N}$ we have $\inf _{u: \mathbb{R}^{N} \rightarrow \mathbb{R}}\left[X_{T}-u\left(W_{T}^{N}, W_{\frac{2 T}{N}}, \ldots, W_{T}\right)\right] \geq c \cdot N^{-1 n n}\left\{1, \frac{28}{\beta^{2}}\right\}$. measurable

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\inf _{\substack{u: \mathbb{R}^{N} \rightarrow \mathbb{R} \\ \text { measurable }}} \mathbb{E}\left[\left|X_{T}-u\left(W_{\frac{T}{N}}, W_{\frac{2 T}{N}}, \ldots, W_{T}\right)\right|\right] \geq c \cdot N^{-\min \left\{1, \frac{2 \delta}{\beta^{2}}\right\}}
$$

Nonlinear PDEs: Deep (2)BSDE method

Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$, normal filtration $\mathbb{T}$ generated by $W$, continuous and adapted $\gamma:[0, T] \times \Omega \rightarrow \mathbb{R}$ and $Z:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ such that $\forall t \in[0, T] \mathbb{P}$-a.s. :

$$
\begin{equation*}
Y_{t}=g\left(\xi+W_{T}\right)+\int_{t}^{T} f\left(Y_{s}, Z_{s}\right) d s-\int_{t}^{T}\left\langle Z_{s}, d W_{s}\right\rangle_{\mathbb{R}^{d}} \tag{BSDE}
\end{equation*}
$$

Under suitable assumptions (Pardoux \& Peng $1990 \ldots$ ) it holds $\forall t \in[0, T] \mathbb{P}$-a.s. :

$$
Y_{t}=u\left(t, \xi+W_{t}\right) \in \mathbb{R} \quad \text { and } \quad Z_{t}=\left(\nabla_{x} u\right)\left(t_{t} \xi+W_{t}\right) \in \mathbb{R}^{d} .
$$

Hence, $\forall t \in[0, T] \mathbb{P}$-a.s. :
$Y_{t}=g\left(\xi+W_{T}\right)+\int_{t}^{T} f\left(Y_{s},\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right)\right) d s-\int_{t}^{T}\left\langle\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right), d W_{s}\right\rangle_{\mathbb{R}^{d}}$. In particular, $\forall t_{1}, t_{2} \in[0, T]$ with $t_{1} \leq t_{2}$ it holds $\mathbb{P}$-a.s. that

$$
Y_{t_{2}}=Y_{t_{1}}-\int_{t_{1}}^{t_{2}} f\left(Y_{s},\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right)\right) d s+\int_{t_{1}}^{t_{2}}\left\langle\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right), d W_{s}\right\rangle_{\mathbb{R}^{d}}
$$

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
$Y_{t_{t+1}} \approx$
$Y_{t_{n}}-f\left(Y_{t_{n}},\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}}$.

Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$,
normal filtration $\mathbb{F}$ generated by $W$, continuous and adapted $Y:[0, T] \times \Omega \rightarrow \mathbb{R}$ and $Z:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ such that $\forall t \in[0, T] \mathbb{P}$-a.s.

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$$

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Under suitable assumptions (Pardoux \& Peng $1990 \ldots$ ) it holds $\forall t \in[0, T] \mathbb{P}$-a.s.:

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$$

Hence, $\forall t \in[0, T] \mathbb{P}$-a.s.
$Y_{t}=g\left(\xi+W_{T}\right)+\int_{t}^{T} f\left(Y_{s},\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right)\right) d s-\int_{t}^{T}\left\langle\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right), d W_{s}\right)_{\mathbb{R}^{d}}$.
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Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$, normal filtration $\mathbb{F}$ generated by $W$, continuous and adapted $Y:[0, T] \times \Omega \rightarrow \mathbb{R}$ and $Z:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ such that $\forall t \in[0, T] \mathbb{P}$-a.s.

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Hence, $\forall t \in[0, T] \mathbb{P}$-a.s.
$\gamma_{t}=g\left(\xi+W_{T}\right)+\int_{t}^{T} f\left(\gamma_{s},\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right)\right) d s-\int_{t}^{T}\left\langle\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right), d W_{s}\right)_{\mathbb{R}^{d}}$
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$Y_{t_{2}}=Y_{t_{1}}-\int_{t_{1}}^{t_{2}} f\left(Y_{s},\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right)\right) d s+\int_{t_{1}}^{t_{2}}\left\langle\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right), d W_{s}\right)_{\mathbb{R}^{d}}$. Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
$\square$

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$$
\begin{equation*}
Y_{t}=g\left(\xi+W_{T}\right)+\int_{t}^{T} f\left(Y_{s}, Z_{s}\right) d s-\int_{t}^{T}\left\langle Z_{s}, d W_{s}\right\rangle_{\mathbb{R}^{d}} \tag{BSDE}
\end{equation*}
$$

Under suitable assumptions (Pardoux \& Peng $1990 \ldots$ ) it holds $\forall t \in[0, T] \mathbb{P}$-a.s. :

$$
Y_{t}=u\left(t, \xi+W_{t}\right) \in \mathbb{R} \quad \text { and } \quad Z_{t}=\left(\nabla_{x} u\right)\left(t, \xi+W_{t}\right) \in \mathbb{R}^{d}
$$

Hence, $\forall t \in[0, T] \mathbb{P}$-a.s. :
$Y_{t}=g\left(\xi+W_{T}\right)+\int_{t}^{T} f\left(Y_{s},\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right)\right) d s-\int_{t}^{T}\left\langle\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right), d W_{s}\right\rangle_{\mathbb{R}^{d}}$. In particular, $\forall t_{1}, t_{2} \in[0, T]$ with $t_{1} \leq t_{2}$ it holds $\mathbb{P}$-a.s. that

$$
Y_{t_{2}}=Y_{t_{1}}-\int_{t_{1}}^{t_{2}} f\left(Y_{s},\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right)\right) d s+\int_{t_{1}}^{t_{2}}\left\langle\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right), d W_{s}\right\rangle_{\mathbb{R}^{d}}
$$

Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$, normal filtration $\mathbb{F}$ generated by $W$, continuous and adapted $Y:[0, T] \times \Omega \rightarrow \mathbb{R}$ and $Z:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ such that $\forall t \in[0, T] \mathbb{P}$-a.s. :

$$
\begin{equation*}
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$$

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$

Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$, normal filtration $\mathbb{F}$ generated by $W$, continuous and adapted $Y:[0, T] \times \Omega \rightarrow \mathbb{R}$ and $Z:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ such that $\forall t \in[0, T] \mathbb{P}$-a.s. :

$$
\begin{equation*}
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Hence, $\forall t \in[0, T] \mathbb{P}$-a.s. :
$Y_{t}=g\left(\xi+W_{T}\right)+\int_{t}^{T} f\left(Y_{s},\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right)\right) d s-\int_{t}^{T}\left\langle\left(\nabla_{x} u\right)\left(s, \xi+W_{s}\right), d W_{s}\right\rangle_{\mathbb{R}^{d}}$. In particular, $\forall t_{1}, t_{2} \in[0, T]$ with $t_{1} \leq t_{2}$ it holds $\mathbb{P}$-a.s. that

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$$

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
$Y_{t_{n+1}} \approx$
$Y_{t_{n}}-f\left(Y_{t_{n}},\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}}$.

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that

$$
\begin{aligned}
& Y_{t_{n+1}} \approx \\
& Y_{t_{n}}-f\left(Y_{t_{n}},\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}} .
\end{aligned}
$$

Consider $\rho=1+3 \mathrm{Nd}(d+1)$, let $\mathcal{R}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{d}\right)$
$\forall \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}, v \in \mathbb{N}_{0}, k, l \in \mathbb{N}$ with $v+k(I+1) \leq \rho$ let $A_{k, l}^{\theta, v}: \mathbb{R}^{\prime} \rightarrow \mathbb{R}^{k}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{l}\right)$
$\left(\begin{array}{cccc}\theta_{v+1} & \theta_{v+2} & \ldots & \theta_{v+l} \\ \theta_{v+l+1} & \theta_{v+l+2} & \ldots & \theta_{v+2 l} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{v+(k-1) l+1} & \theta_{v+(k-1) l+2} & \ldots & \theta_{v+k l}\end{array}\right)\left(\begin{array}{c}x_{1} \\ x_{2} \\ \vdots \\ x_{l}\end{array}\right)+\left(\begin{array}{c}\theta_{v+k l+1} \\ \theta_{v+k l+2} \\ \vdots \\ \theta_{v+k l+k}\end{array}\right)$
$\forall \theta \in \mathbb{R}^{\rho}, n \in\{0,1, \ldots, N\}$ let $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy

$\square$

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
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Consider $\rho=1+3 \mathrm{Nd}(d+1)$, let $\mathcal{R}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{d}\right)$

$$
\mathcal{R}(x)=\left(\max \left\{x_{1}, 0\right\}, \ldots, \max \left\{x_{d}, 0\right\}\right),
$$

$\forall \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}, v \in \mathbb{N}_{0}, k, l \in \mathbb{N}$ with $v+k(I+1) \leq \rho$ let $A_{k, l}^{\theta, v}: \mathbb{R}^{\prime} \rightarrow \mathbb{R}^{k}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{l}\right):$ $\left(\begin{array}{cccc}\theta_{v+1} & \theta_{v+2} & \ldots & \theta_{v+l} \\ \theta_{v+l+1} & \theta_{v+l+2} & \ldots & \theta_{v+2 l} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{v+(k-1) l+1} & \theta_{v+(k-1) l+2} & \cdots & \theta_{v+k l}\end{array}\right)\left(\begin{array}{c}x_{1} \\ x_{2} \\ \vdots \\ x_{l}\end{array}\right)+\left(\begin{array}{c}\theta_{v+k l+1} \\ \theta_{v+k l+2} \\ \vdots \\ \theta_{v+k l+k}\end{array}\right)$ $\forall \theta \in \mathbb{R}^{\rho}, n \in\{0,1, \ldots, N\}$ let $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy


Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
$Y_{t_{n+1}} \approx$
$Y_{t_{n}}-f\left(Y_{t_{n}},\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}}$.
Consider $\rho=1+3 \mathrm{Nd}(d+1)$, let $\mathcal{R}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy

$$
\mathcal{R}(x)=\left(\max \left\{x_{1}, 0\right\},\right.
$$

$\left.\max \left\{x_{d}, 0\right\}\right)$,
$\forall \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}, v \in \mathbb{N}_{0}, k, l \in \mathbb{N}$ with $v+k(I+1) \leq \rho$ let $A_{k, l}^{\theta, v}: \mathbb{R}^{\prime} \rightarrow \mathbb{R}^{k}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{l}\right)$ :

$\forall \theta \in \mathbb{R}^{\rho}, n \in\{0,1, \ldots, N\}$ let $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy


Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that

$$
\begin{aligned}
& Y_{t_{n+1}} \approx \\
& Y_{t_{n}}-f\left(Y_{t_{n}},\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}} .
\end{aligned}
$$

Consider $\rho=1+3 \mathrm{Nd}(d+1)$, let $\mathcal{R}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{d}\right)$ :

$$
\mathcal{R}(x)=\left(\max \left\{x_{1}, 0\right\}, \ldots, \max \left\{x_{d}, 0\right\}\right),
$$

$\forall \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}, v \in \mathbb{N}_{0}, k, l \in \mathbb{I}$
$A_{k, l}^{\theta, v}: \mathbb{R}^{\prime} \rightarrow \mathbb{R}^{k}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{l}\right):$

$\forall \theta \in \mathbb{R}^{\rho}, n \in\{0,1, \ldots, N\}$ let $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy


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$$
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\end{aligned}
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Consider $\rho=1+3 \mathrm{Nd}(d+1)$, let $\mathcal{R}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{d}\right)$ :

$$
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$$

$\forall \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}, v \in \mathbb{N}_{0}, k, l \in \mathbb{N}$ with $v+k(l+1) \leq \rho$ let
$\square$

$\square$

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that

$$
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\end{aligned}
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$\square$

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that $Y_{t_{n+1}} \approx$
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$\forall \theta \in \mathbb{R}^{\rho}, n \in\{0,1, \ldots, N\}$ let $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy

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\end{aligned}
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$$
A_{k, l}^{\theta, v}(x)=\left(\begin{array}{cccc}
\theta_{v+1} & \theta_{v+2} & \ldots & \theta_{v+l} \\
\theta_{v+l+1} & \theta_{v+l+2} & \ldots & \theta_{v+2 l} \\
\vdots & \vdots & \vdots & \vdots \\
\theta_{v+(k-1) l+1} & \theta_{v+(k-1) l+2} & \ldots & \theta_{v+k l}
\end{array}\right)\left(\begin{array}{c}
x_{1} \\
x_{2} \\
\vdots \\
x_{l}
\end{array}\right)+
$$

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
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Consider $\rho=1+3 \operatorname{Nd}(d+1)$, let $\mathcal{R}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{d}\right)$ :

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\mathcal{R}(x)=\left(\max \left\{x_{1}, 0\right\}, \ldots, \max \left\{x_{d}, 0\right\}\right),
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$\forall \theta \in \mathbb{R}^{\rho}, n \in\{0,1, \ldots, N\}$ let $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy
and $\forall \theta \in \mathbb{R}^{\rho}$ let $\mathcal{Y}^{\theta}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}$ satisfy $\mathcal{Y}_{0}^{\theta}=\theta_{1}$ and

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
$Y_{t_{n+1}} \approx$
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Consider $\rho=1+3 \operatorname{Nd}(d+1)$, let $\mathcal{R}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{d}\right)$ :

$$
\mathcal{R}(x)=\left(\max \left\{x_{1}, 0\right\}, \ldots, \max \left\{x_{d}, 0\right\}\right),
$$

$\forall \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}, v \in \mathbb{N}_{0}, k, l \in \mathbb{N}$ with $v+k(I+1) \leq \rho$ let $A_{k, l}^{\theta, v}: \mathbb{R}^{\prime} \rightarrow \mathbb{R}^{k}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{l}\right)$ :
$A_{k, l}^{\theta, v}(x)=\left(\begin{array}{cccc}\theta_{v+1} & \theta_{v+2} & \ldots & \theta_{v+l} \\ \theta_{v+l+1} & \theta_{v+l+2} & \ldots & \theta_{v+2 l} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{v+(k-1) l+1} & \theta_{v+(k-1) l+2} & \ldots & \theta_{v+k l}\end{array}\right)\left(\begin{array}{c}x_{1} \\ x_{2} \\ \vdots \\ x_{l}\end{array}\right)+\left(\begin{array}{c}\theta_{v+k l+1} \\ \theta_{v+k l+2} \\ \vdots \\ \theta_{v+k l+k}\end{array}\right)$
$\forall \theta \in \mathbb{R}^{\rho}, n \in\{0,1, \ldots, N\}$

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
$Y_{t_{n+1}} \approx$
$Y_{t_{n}}-f\left(Y_{t_{n}},\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}}$.
Consider $\rho=1+3 \operatorname{Nd}(d+1)$, let $\mathcal{R}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{d}\right)$ :

$$
\mathcal{R}(x)=\left(\max \left\{x_{1}, 0\right\}, \ldots, \max \left\{x_{d}, 0\right\}\right),
$$

$\forall \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}, v \in \mathbb{N}_{0}, k, l \in \mathbb{N}$ with $v+k(I+1) \leq \rho$ let $A_{k, l}^{\theta, v}: \mathbb{R}^{\prime} \rightarrow \mathbb{R}^{k}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{l}\right)$ :
$A_{k, l}^{\theta, v}(x)=\left(\begin{array}{cccc}\theta_{v+1} & \theta_{v+2} & \ldots & \theta_{v+l} \\ \theta_{v+l+1} & \theta_{v+l+2} & \ldots & \theta_{v+2 l} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{v+(k-1) l+1} & \theta_{v+(k-1) l+2} & \ldots & \theta_{v+k l}\end{array}\right)\left(\begin{array}{c}x_{1} \\ x_{2} \\ \vdots \\ x_{l}\end{array}\right)+\left(\begin{array}{c}\theta_{v+k l+1} \\ \theta_{v+k l+2} \\ \vdots \\ \theta_{v+k l+k}\end{array}\right)$
$\forall \theta \in \mathbb{R}^{\rho}, n \in\{0,1, \ldots, N\}$ let $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
$Y_{t_{n+1}} \approx$
$Y_{t_{n}}-f\left(Y_{t_{n}},\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}}$.
Consider $\rho=1+3 \mathrm{Nd}(d+1)$, let $\mathcal{R}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{d}\right)$ :

$$
\mathcal{R}(x)=\left(\max \left\{x_{1}, 0\right\}, \ldots, \max \left\{x_{d}, 0\right\}\right),
$$

$\forall \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}, v \in \mathbb{N}_{0}, k, l \in \mathbb{N}$ with $v+k(I+1) \leq \rho$ let $A_{k, l}^{\theta, v}: \mathbb{R}^{\prime} \rightarrow \mathbb{R}^{k}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{l}\right):$
$A_{k, l}^{\theta, v}(x)=\left(\begin{array}{cccc}\theta_{v+1} & \theta_{v+2} & \ldots & \theta_{v+l} \\ \theta_{v+l+1} & \theta_{v+l+2} & \ldots & \theta_{v+2 l} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{v+(k-1) l+1} & \theta_{v+(k-1) /+2} & \ldots & \theta_{v+k l}\end{array}\right)\left(\begin{array}{c}x_{1} \\ x_{2} \\ \vdots \\ x_{l}\end{array}\right)+\left(\begin{array}{c}\theta_{v+k l+1} \\ \theta_{v+k l+2} \\ \vdots \\ \theta_{v+k l+k}\end{array}\right)$
$\forall \theta \in \mathbb{R}^{\rho}, n \in\{0,1, \ldots, N\}$ let $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy

$$
\mathcal{V}_{n}^{\theta}=A_{d, d}^{\theta, 1+3 n d(d+1)} \circ \mathcal{R} \circ A_{d, d}^{\theta, 1+(3 n+1) d(d+1)} \circ \mathcal{R} \circ A_{d, d}^{\theta, 1+(3 n+2) d(d+1)}
$$

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
$Y_{t_{n+1}} \approx$
$Y_{t_{n}}-f\left(Y_{t_{n}},\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}}$.
Consider $\rho=1+3 \operatorname{Nd}(d+1)$, let $\mathcal{R}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{d}\right)$ :

$$
\mathcal{R}(x)=\left(\max \left\{x_{1}, 0\right\}, \ldots, \max \left\{x_{d}, 0\right\}\right),
$$

$\forall \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}, v \in \mathbb{N}_{0}, k, l \in \mathbb{N}$ with $v+k(I+1) \leq \rho$ let $A_{k, l}^{\theta, v}: \mathbb{R}^{\prime} \rightarrow \mathbb{R}^{k}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{l}\right)$ :
$A_{k, l}^{\theta, v}(x)=\left(\begin{array}{cccc}\theta_{v+1} & \theta_{v+2} & \ldots & \theta_{v+l} \\ \theta_{v+l+1} & \theta_{v+l+2} & \ldots & \theta_{v+2 l} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{v+(k-1) l+1} & \theta_{v+(k-1) l+2} & \ldots & \theta_{v+k l}\end{array}\right)\left(\begin{array}{c}x_{1} \\ x_{2} \\ \vdots \\ x_{l}\end{array}\right)+\left(\begin{array}{c}\theta_{v+k l+1} \\ \theta_{v+k l+2} \\ \vdots \\ \theta_{v+k l+k}\end{array}\right)$
$\forall \theta \in \mathbb{R}^{\rho}, n \in\{0,1, \ldots, N\}$ let $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy

$$
\mathcal{V}_{n}^{\theta}=A_{d, d}^{\theta, 1+3 n d(d+1)} \circ \mathcal{R} \circ A_{d, d}^{\theta, 1+(3 n+1) d(d+1)} \circ \mathcal{R} \circ A_{d, d}^{\theta, 1+(3 n+2) d(d+1)}
$$

and $\forall \theta \in \mathbb{R}^{\rho}$ let $\mathcal{Y}^{\theta}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}$ satisfy $\mathcal{Y}_{0}^{\theta}=\theta_{1}$

Consider $N \in \mathbb{N}$ and $0=t_{0}<t_{1}<\ldots<t_{N}=T$ and observe that
$Y_{t_{n+1}} \approx$
$Y_{t_{n}}-f\left(Y_{t_{n}},\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\left(\nabla_{x} u\right)\left(t_{n}, \xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}}$.
Consider $\rho=1+3 \operatorname{Nd}(d+1)$, let $\mathcal{R}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{d}\right)$ :

$$
\mathcal{R}(x)=\left(\max \left\{x_{1}, 0\right\}, \ldots, \max \left\{x_{d}, 0\right\}\right),
$$

$\forall \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}, v \in \mathbb{N}_{0}, k, l \in \mathbb{N}$ with $v+k(I+1) \leq \rho$ let $A_{k, l}^{\theta, v}: \mathbb{R}^{\prime} \rightarrow \mathbb{R}^{k}$ satisfy $\forall x=\left(x_{1}, \ldots, x_{l}\right)$ :
$A_{k, l}^{\theta, v}(x)=\left(\begin{array}{cccc}\theta_{v+1} & \theta_{v+2} & \ldots & \theta_{v+l} \\ \theta_{v+l+1} & \theta_{v+l+2} & \ldots & \theta_{v+2 l} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{v+(k-1) l+1} & \theta_{v+(k-1) /+2} & \ldots & \theta_{v+k l}\end{array}\right)\left(\begin{array}{c}x_{1} \\ x_{2} \\ \vdots \\ x_{l}\end{array}\right)+\left(\begin{array}{c}\theta_{v+k l+1} \\ \theta_{v+k l+2} \\ \vdots \\ \theta_{v+k l+k}\end{array}\right)$
$\forall \theta \in \mathbb{R}^{\rho}, n \in\{0,1, \ldots, N\}$ let $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfy

$$
\mathcal{V}_{n}^{\theta}=A_{d, d}^{\theta, 1+3 n d(d+1)} \circ \mathcal{R} \circ A_{d, d}^{\theta, 1+(3 n+1) d(d+1)} \circ \mathcal{R} \circ A_{d, d}^{\theta, 1+(3 n+2) d(d+1)}
$$

and $\forall \theta \in \mathbb{R}^{\rho}$ let $\mathcal{Y}^{\theta}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}$ satisfy $\mathcal{Y}_{0}^{\theta}=\theta_{1}$ and
$\mathcal{Y}_{n+1}^{\theta}=\mathcal{Y}_{n}^{\theta}-f\left(\mathcal{Y}_{n}^{\theta}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}}$.
$\forall \theta \in \mathbb{R}^{\rho}$ let $\mathcal{Y}^{\theta}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}$ satisfy $\mathcal{Y}_{0}^{\theta}=\theta_{1}$ and
$\mathcal{\nu}_{n+1}^{\theta}=\mathcal{Y}_{n}^{\theta}-f\left(\mathcal{\nu}_{n}^{\theta}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}}$.
We suggest to minimize

$$
\mathbb{R}^{P} \ni \theta \mapsto \mathbb{E}\left[\left|\mathcal{Y}_{N}^{\theta}-g\left(\xi+W_{T}\right)\right|^{2}\right] \in[0, \infty]
$$

Consider stochastic gradient descent-type approximations

$$
\theta=\left(\Theta^{(1)}, \ldots \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{p}
$$

associated to (Optimization problem). We suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that

$$
\Theta_{m}^{(1)} \approx u(0, \xi)
$$

$\forall \theta \in \mathbb{R}^{\rho}$ let $\mathcal{Y}^{\theta}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}$ satisfy $\mathcal{Y}_{0}^{\theta}=\theta_{1}$ and

$$
\mathcal{Y}_{n+1}^{\theta}=\mathcal{Y}_{n}^{\theta}-f\left(\mathcal{Y}_{n}^{\theta}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}} .
$$

## We suggest to minimize

$$
\mathbb{R}^{\rho} \ni \theta \mapsto \mathbb{E}\left[\left|\mathcal{Y}_{N}^{\theta}-g\left(\xi+W_{T}\right)\right|^{2}\right] \in[0, \infty] .
$$

## Consider stochastic gradient descent-type approximations


associated to (Optimization problem). We suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that
$\forall \theta \in \mathbb{R}^{\rho}$ let $\mathcal{Y}^{\theta}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}$ satisfy $\mathcal{Y}_{0}^{\theta}=\theta_{1}$ and $\mathcal{Y}_{n+1}^{\theta}=\mathcal{Y}_{n}^{\theta}-f\left(\mathcal{Y}_{n}^{\theta}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}}$.

We suggest to minimize

$$
\mathbb{R}^{\rho} \ni \theta \mapsto \mathbb{E}\left[\left|\mathcal{Y}_{N}^{\theta}-g\left(\xi+W_{T}\right)\right|^{2}\right] \in[0, \infty]
$$

(Optimization problem)

## Consider stochastic gradient descent-type approximations


associated to (Optimization problem). We suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that
$\forall \theta \in \mathbb{R}^{\rho}$ let $\mathcal{Y}^{\theta}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}$ satisfy $\mathcal{Y}_{0}^{\theta}=\theta_{1}$ and $\mathcal{Y}_{n+1}^{\theta}=\mathcal{Y}_{n}^{\theta}-f\left(\mathcal{Y}_{n}^{\theta}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}}$.

We suggest to minimize

$$
\mathbb{R}^{\rho} \ni \theta \mapsto \mathbb{E}\left[\left|\mathcal{Y}_{N}^{\theta}-g\left(\xi+W_{T}\right)\right|^{2}\right] \in[0, \infty]
$$

(Optimization problem)
Consider stochastic gradient descent-type approximations

$$
\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}
$$

associated to (Optimization problem). We suggest for sufficiently large N, p, m $\in \mathbb{N}$ that
$\forall \theta \in \mathbb{R}^{\rho}$ let $\mathcal{Y}^{\theta}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}$ satisfy $\mathcal{Y}_{0}^{\theta}=\theta_{1}$ and $\mathcal{Y}_{n+1}^{\theta}=\mathcal{Y}_{n}^{\theta}-f\left(\mathcal{Y}_{n}^{\theta}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}\right), W_{t_{n+1}}-W_{t_{n}}\right\rangle_{\mathbb{R}^{d}}$.

We suggest to minimize

$$
\mathbb{R}^{\rho} \ni \theta \mapsto \mathbb{E}\left[\left|\mathcal{Y}_{N}^{\theta}-g\left(\xi+W_{T}\right)\right|^{2}\right] \in[0, \infty]
$$

(Optimization problem)
Consider stochastic gradient descent-type approximations

$$
\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}
$$

associated to (Optimization problem). We suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that

$$
\Theta_{m}^{(1)} \approx u(0, \xi)
$$

(Deep BSDE method)

## Consider

 $0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$,$0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\nu_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$, $0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function $\mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$ satisfying $\mathcal{Y}_{0}^{\theta, m}=\theta_{1}$ and $\forall n=0,1, \ldots, N-1$. $\mathcal{V}_{n+1}^{\theta, m}=\mathcal{Y}_{n}^{\theta, m}-f\left(\mathcal{Y}_{n}^{\theta, m}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle_{\mathbb{R}}$ for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\nu_{N}^{\theta, m}-g\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}, \omega \in \Omega: \quad \phi^{m}(\theta, \omega)=\left(\nabla_{\theta} \phi^{m}\right)(\theta, \omega),
$$

and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\forall m \in \mathbb{N}: \quad \Theta_{m}=\Theta_{m-1}-\gamma \cdot \Phi^{m}\left(\Theta_{m-1}\right) \text {. }
$$

We suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that $\Theta_{m}^{(1)} \approx u(0, \xi)$.

Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$,
$0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\nu_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$,
$0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function
$\mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$ satisfying $\mathcal{Y}_{0}^{\theta, m}=\theta_{1}$ and $\forall n=0,1, \ldots, N-1$.
$\mathcal{Y}_{n+1}^{\theta, m}=\mathcal{Y}_{n}^{\theta, m}-f\left(\mathcal{Y}_{n}^{\theta, m}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle_{\mathbb{R}}$
for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\mathcal{\nu}_{N}^{\theta, m}-g\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}, \omega \in \Omega: \quad \phi^{m}(\theta, \omega)=\left(\nabla_{\theta} \phi^{m}\right)(\theta, \omega),
$$

and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
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Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$,
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$0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function
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for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\vartheta_{N}^{\theta \cdot m}-g^{\prime}\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}, \omega \in \Omega: \quad \phi^{m}(\theta, \omega)=\left(\nabla_{\theta} \phi^{m}\right)(\theta, \omega),
$$

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Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$.
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$0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function
$\mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$ satisfying $\mathcal{Y}_{0}^{\theta, m}=\theta_{1}$ and $\forall n=0,1, \ldots, N-1$.
$\mathcal{V}_{n+1}^{\theta, m}=\mathcal{V}_{n}^{\theta, m}-f\left(\mathcal{V}_{n}^{\theta, m}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle$
for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\vartheta_{N}^{\theta \cdot m}-g^{\prime}\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}, \omega \in \Omega: \quad \phi^{m}(\theta, \omega)=\left(\nabla_{\theta} \phi^{m}\right)(\theta, \omega),
$$

and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\forall m \in \mathbb{N}: \quad \Theta_{m}=\Theta_{m-1}-\gamma \cdot \Phi^{m}\left(\Theta_{m-1}\right)
$$

We suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that $\Theta_{m}^{(1)} \approx u(0, \xi)$.

Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$
$0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$, $0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function $\mathcal{\nu}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$ satisfying $\mathcal{V}_{0}^{\theta, m}=\theta_{1}$ and $\forall n=0,1, \ldots, N-1$.

$$
=\mathcal{Y}_{n}^{\theta, m}-f\left(\mathcal{Y}_{n}^{\theta, m}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle_{1}
$$

for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\vartheta_{N}^{\theta \cdot m}-g^{\prime}\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}, \omega \in \Omega: \quad \Phi^{m}(\theta, \omega)=\left(\nabla_{\theta} \phi^{m}\right)(\theta, \omega),
$$

and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\forall m \in \mathbb{N}: \quad \Theta_{m}=\Theta_{m-1}-\gamma \cdot \Phi^{m}\left(\Theta_{m-1}\right) .
$$

We suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that $\Theta_{m}^{(1)} \approx u(0, \xi)$.

Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$, $0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$, $0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function $\mathcal{V}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$ satisfying $\mathcal{V}_{0}^{\theta, m}=\theta_{1}$ and $\forall n=0,1, \ldots, N-1$. $=\mathcal{Y}_{n}^{\theta, m}-f\left(\mathcal{Y}_{n}^{\theta, m}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle$ for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\mathcal{Y}_{N}^{\theta, m}-g\left(\xi+W_{T}^{m}\right)\right|^{2},
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

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$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\vartheta_{N}^{\theta \cdot m}-g^{\prime}\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}, \omega \in \Omega: \quad \phi^{m}(\theta, \omega)=\left(\nabla_{\theta} \phi^{m}\right)(\theta, \omega),
$$

and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

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Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$, $0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian
motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\nu_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$,
$0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function
$\nu^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$ satisfying $\nu_{0}^{\theta} m=\theta_{1}$ and $\forall n=0,1, \ldots, N-1:$
$\theta_{n+1}^{\theta_{n}}=\nu_{n}^{\theta}, m-f\left(\nu_{n}^{\theta, m}, \nu_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\nu_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle$
for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\vartheta_{N}^{\theta, m}-g^{\prime}\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}, \omega \in \Omega: \quad \phi^{m}(\theta, \omega)=\left(\nabla_{\theta} \phi^{m}\right)(\theta, \omega),
$$

and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying


Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$, $0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$,

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}, \omega \in \Omega: \quad \phi^{m}(\theta, \omega)=\left(\nabla_{\theta} \phi^{m}\right)(\theta, \omega),
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for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{n}: \quad \phi^{m}(\theta)=\left|\gamma_{N}^{\theta \cdot m}-g^{\prime}\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying


Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$, $0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$, $0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function
for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying
for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying
and $\Theta=\left(\Theta^{(1)}\right.$


Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$, $0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$, $0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function $\mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$
for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\nu_{N}^{\theta \cdot m}-g^{\prime}\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying


Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$, $0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$, $0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function $\mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$ satisfying $\mathcal{Y}_{0}^{\theta, m}=\theta_{1}$
for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying
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and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$, $0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$, $0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function $\mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$ satisfying $\mathcal{Y}_{0}^{\theta, m}=\theta_{1}$ and $\forall n=0,1, \ldots, N-1$ :
for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying
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and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

$$
\begin{aligned}
& \text { Consider } T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}, \\
& 0=t_{0}<t_{1}<\ldots<t_{N}=T, \text { probability space }(\Omega, \mathcal{F}, \mathbb{P}) \text {, independent Brownian } \\
& \text { motions } W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0} \text {, functions } \mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}, \\
& 0 \leq n \leq N, \text { for every } m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho} \text { a function } \\
& \mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k} \text { satisfying } \mathcal{Y}_{0}^{\theta, m}=\theta_{1} \text { and } \forall n=0,1, \ldots, N-1: \\
& \mathcal{Y}_{n+1}^{\theta, m}=\mathcal{Y}_{n}^{\theta, m}-f\left(\mathcal{Y}_{n}^{\theta, m}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle_{\mathbb{R}^{d}},
\end{aligned}
$$

for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying
for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying


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& \text { Consider } T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}, \\
& 0=t_{0}<t_{1}<\ldots<t_{N}=T, \text { probability space }(\Omega, \mathcal{F}, \mathbb{P}) \text {, independent Brownian } \\
& \text { motions } W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0} \text {, functions } \mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}, \\
& 0 \leq n \leq N, \text { for every } m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho} \text { a function } \\
& \mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k} \text { satisfying } \mathcal{Y}_{0}^{\theta, m}=\theta_{1} \text { and } \forall n=0,1, \ldots, N-1: \\
& \mathcal{Y}_{n+1}^{\theta, m}=\mathcal{Y}_{n}^{\theta, m}-f\left(\mathcal{Y}_{n}^{\theta, m}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle_{\mathbb{R}^{d}},
\end{aligned}
$$

for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying
for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying


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$\mathcal{Y}_{n+1}^{\theta, m}=\mathcal{Y}_{n}^{\theta, m}-f\left(\mathcal{Y}_{n}^{\theta, m}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle_{\mathbb{R}^{d}}$,
for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\mathcal{Y}_{N}^{\theta, m}-g\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying
and $\Theta=\left(\Theta^{(1)}, \ldots, \Theta^{(\rho)}\right): \mathbb{N}_{0} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$, $0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$,
$0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function $\mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$ satisfying $\mathcal{Y}_{0}^{\theta, m}=\theta_{1}$ and $\forall n=0,1, \ldots, N-1$ :
$\mathcal{Y}_{n+1}^{\theta, m}=\mathcal{Y}_{n}^{\theta, m}-f\left(\mathcal{Y}_{n}^{\theta, m}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle_{\mathbb{R}^{d}}$,
for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

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\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\mathcal{Y}_{N}^{\theta, m}-g\left(\xi+W_{T}^{m}\right)\right|^{2}
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying


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$\mathcal{Y}_{n+1}^{\theta, m}=\mathcal{Y}_{n}^{\theta, m}-f\left(\mathcal{Y}_{n}^{\theta, m}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle_{\mathbb{R}^{d}}$,
for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\mathcal{Y}_{N}^{\theta, m}-g\left(\xi+W_{T}^{m}\right)\right|^{2},
$$

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$$
\forall \theta \in \mathbb{R}^{\rho}, \omega \in \Omega: \quad \Phi^{m}(\theta, \omega)=\left(\nabla_{\theta} \phi^{m}\right)(\theta, \omega)
$$

and $\Theta=\left(\Theta^{(1)}\right.$,


Consider $T, \gamma>0, d, \rho, N \in \mathbb{N}, \xi \in \mathbb{R}^{d}, f: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$, $0=t_{0}<t_{1}<\ldots<t_{N}=T$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent Brownian motions $W^{m}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, m \in \mathbb{N}_{0}$, functions $\mathcal{V}_{n}^{\theta}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \theta \in \mathbb{R}^{\rho}$,
$0 \leq n \leq N$, for every $m \in \mathbb{N}_{0}, \theta=\left(\theta_{1}, \ldots, \theta_{\rho}\right) \in \mathbb{R}^{\rho}$ a function $\mathcal{Y}^{\theta, m}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{k}$ satisfying $\mathcal{Y}_{0}^{\theta, m}=\theta_{1}$ and $\forall n=0,1, \ldots, N-1$ :
$\mathcal{Y}_{n+1}^{\theta, m}=\mathcal{Y}_{n}^{\theta, m}-f\left(\mathcal{Y}_{n}^{\theta, m}, \mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right)\right)\left(t_{n+1}-t_{n}\right)+\left\langle\mathcal{V}_{n}^{\theta}\left(\xi+W_{t_{n}}^{m}\right), W_{t_{n+1}}^{m}-W_{t_{n}}^{m}\right\rangle_{\mathbb{R}^{d}}$,
for every $m \in \mathbb{N}_{0}$ a function $\phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}$ satisfying

$$
\forall \theta \in \mathbb{R}^{\rho}: \quad \phi^{m}(\theta)=\left|\mathcal{Y}_{N}^{\theta, m}-g\left(\xi+W_{T}^{m}\right)\right|^{2},
$$

for every $m \in \mathbb{N}_{0}$ a function $\Phi^{m}: \mathbb{R}^{\rho} \times \Omega \rightarrow \mathbb{R}^{\rho}$ satisfying

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\forall \theta \in \mathbb{R}^{\rho}, \omega \in \Omega: \quad \Phi^{m}(\theta, \omega)=\left(\nabla_{\theta} \phi^{m}\right)(\theta, \omega)
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We suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that

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We suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that $\Theta_{m}^{(1)} \approx u(0, \xi)$.

## Numerical simulations

> Implementations in Python using TensorFlow on a Macpook Pro 2.9 GHz (Intel i5, 16 GB RAM)

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## 100-dimensional Allen-Cahn equation

Consider

$$
\frac{\partial u}{\partial t}(t, x)=\left(\Delta_{x} u\right)(t, x)+u(t, x)-[u(t, x)]^{3}
$$

(Allen-Cahn)
with $u(0, x)=\frac{1}{\left(2+0.4\|x\|^{2}\right)}$ for $t \in\left[0, \frac{3}{10}\right], x \in \mathbb{R}^{100}$.

(a) Relative $L^{1}$-error for $u\left(\frac{3}{10}, 0\right) \approx 0.0528$

(b) Approximative plot of $u(t, 0), 0 \leq t \leq \frac{3}{10}$

Deep BSDE method ( $N=20, \gamma=\frac{5}{10000}$ ): $L^{1}$-error: $0.3 \%$, Runtime: 647 seconds.

## 100-dimensional Hamiltonian-Jacobi-Bellman equation

Consider

$$
\begin{equation*}
\frac{\partial u}{\partial t}(t, x)+\left(\Delta u_{x}\right)(t, x)=\lambda\left\|\left(\nabla_{x} u\right)(t, x)\right\|^{2} \tag{HJB}
\end{equation*}
$$

with $u(1, x)=\frac{2}{\left(1+\|x\|^{2}\right)}, \lambda \geq 0$ for $t \in[0,1], x \in \mathbb{R}^{100}$.

(a) Relative $L^{1}$-error when $\lambda=1$

(b) Optimal cost against different $\lambda$

Deep BSDE method ( $N=20, \gamma=\frac{1}{100}$ ): $L^{1}$-error: $0.17 \%$, Runtime: 330 seconds.

## 100-dimensional pricing model incorporating default risk

Duffie, Schroder, \& Skiadas 1996, Bender, Schweizer, \& Zhuo 2015:
$\frac{\partial u}{\partial t}(t, x)+\bar{\mu}\left\langle x,\left(\nabla_{x} u\right)(t, x)\right\rangle_{\mathbb{R}^{d}}+\frac{\bar{\sigma}^{2}}{2} \sum_{i=1}^{d}\left|x_{i}\right|^{2} \frac{\partial^{2} u}{\partial x_{i}^{2}}(t, x)-Q(u(t, x)) u(t, x)-R u(t, x)=0$
with $u(1, x)=\min _{1 \leq i \leq 100} x_{i}, \bar{\mu}=R=2 \%, \bar{\sigma}=20 \%$ for $t \in[0,1], x \in \mathbb{R}^{100}$.


Approximations for $u(0,100, \ldots 100) \approx 57.3$ (defaultrisk excluded: $\approx 60.8$ )
Deep BSDE method $\left(N=40, \gamma=\frac{8}{1000}\right): L^{1}$-error: $0.46 \%$, Runtime: 617 seconds.

## Fully-nonlinear PDEs:

- A 100-dimensional Black-Scholes-Barenblatt equation
- Nonlinear expectations of G-Brownian motions in 1 and 100 space-dimensions


## All source codes available on GItHUB or ARXIV:

- E, Han, \& J, Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations. arXiv 2017. Comm. Math. Stat. (2017)
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Outlook: Other PDEs and Proofs!
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100-dimensional pricing model with different interest rates (Bergman 1995)
Consider $\bar{\sigma}=20 \%, R^{\prime}=4 \%, R^{b}=6 \%$ and for $t \in[0,1 / 2], x \in \mathbb{R}^{100}$ :

$$
\frac{\partial u}{\partial t}+\frac{\bar{\sigma}^{2}}{2} \sum_{i=1}^{d}\left|x_{i}\right|^{2} \frac{\partial^{2} u}{\partial x_{i}^{2}}-\min \left\{R^{b}\left(u-\sum_{i=1}^{d} x_{i} \frac{\partial u}{\partial x_{i}}\right), R^{\prime}\left(u-\sum_{i=1}^{d} x_{i} \frac{\partial u}{\partial x_{i}}\right)\right\}=0
$$

with $u(1 / 2, x)=\max \left\{\left[\max _{i} x_{i}\right]-120,0\right\}-2 \max \left\{\left[\max _{i} x_{i}\right]-150,0\right\}$.


Relative $L^{1}$-error for $u(0,100, \ldots 100) \approx 21.299$
Deep BSDE method ( $N=20, \gamma=\frac{1}{200}$ ): $L^{1}$-error: $0.39 \%$, Runtime: 566 seconds.

Pricing with default risk (Duffie, Schroder, \& Skiadas 1996 AAP, Bender, Schweizer, \& Zhuo 2015 MF)

Consider $\delta=\frac{2}{3}, \gamma^{h}=\frac{2}{10}, \gamma^{\prime}=\frac{2}{100}, v^{h}, v^{\prime} \in(0, \infty)$ satisfying $v^{h}<v^{\prime}$ and
$Q(y)=$
$(1-\delta)\left[\gamma^{h} \mathbb{1}_{\left(-\infty, v^{h}\right)}(y)+\gamma^{\prime} \mathbb{1}_{\left[v^{\prime}, \infty\right)}(y)+\left[\frac{\left(\gamma^{h}-\gamma^{\prime}\right)}{\left(v^{h}-v^{\prime}\right)}\left(y-v^{h}\right)+\gamma^{h}\right] \mathbb{1}_{\left[v^{h}, v^{\prime}\right)}(y)\right]$.

- Bender et al. consider $v^{h}=54, v^{\prime}=90$ in the case $d=5$.
- We consider $v^{h}=50, v^{\prime}=70$ in the case $d=100$.

Plot of $\left\|\mathbb{E}\left[X_{T}\right]-\mathbb{E}\left[Y_{N}^{N}\right]\right\|$ for $T=2$ and $N \in\left\{2^{1}, 2^{2}, \ldots, 2^{30}\right\}$.


