Research Highlight:

Duality for Spherical Representations in Exceptional Theta Correspondences

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Let H be a real split Lie group of type $\mathbf{E}_n$ where $n = 6, 7, 8$.

If $n = 6$, we set $H = H^0 \rtimes \mathbb{Z}/2\mathbb{Z}$ where H^0 is the real points of a split, simply connected algebraic group, simply connected group of type E_6. It contains a split dual pair $G \times G'$ is such that $G \cong \mathrm{SL}_3(\mathbb{R}) \rtimes \mathbb{Z}/2\mathbb{Z}$ and G is of the type $\mathbf{G}_2$.

If $n = 7$ or 8, we set H to be the group of real points of a split, simply connected algebraic group of the type $\mathbf{E}_n$. The group H contains a split dual pair $G \times G'$ where G is of the type $\mathbf{G}_2$, while G' is a simply connected group of the type $\mathbf{C}_3$ and $\mathbf{F}_4$ respectively.

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be the Lie algebras of G and G'' respectively, and let K and K' be the maximal compact subgroups of G and G', respectively. Let $\mathbf{V}$ be the Harish-Chandra module of the minimal representation of H. Let V be an irreducible $(\mathfrak{g}, K)$-module and V' be an irreducible $(\mathfrak{g}', K')$-module. We say that V and V' correspond if $V \otimes V'$ is a quotient of $\mathbf{V}$. Let V be an irreducible $(\mathfrak{g}, K)$-module. There is a $(\mathfrak{g}', K')$-module $\Theta(V)$ such that

$$\mathbf{V}/\cap_{\phi} \ker \phi \simeq \Theta(V) \otimes V$$

where the intersection is taken over all $(\mathfrak{g}, K)$-module homomorphisms $\phi\colon \mathbf{V} \to V$. Motivated by the classical dual pair correspondences, it is conjectured that $\Theta(V)$ is a finite length $(\mathfrak{g}', K')$-module with a unique irreducible quotient V', and then conversely, that $\Theta(V')$ is a finite length $(\mathfrak{g}, K)$-module with V as a unique irreducible quotient. We call this a *strong duality*.

Let λ and λ' denote the infinitesimal characters of V and V' respectively. It is known previously that there is an explicit correspondence of the infinitesimal characters. A spherical representation V of G is an irreducible representation in which V^{K° is nonzero. It is a fact that a spherical representation is uniquely determined by its infinitesimal character. We let S_λ denote the spherical representation of G with infinitesimal char-acter λ. Likewise we let $S_{\lambda'}$ to be the spherical representation of G' with infinitesimal character λ $'$.

Our first main result is that if $\Theta(S_{\lambda'}) \neq 0$ then it is a finite length $(\mathfrak{g}, K)$-module with the unique irreducible quotient isomorphic to S_λ. Here λ is the infinitesimal character corresponding to λ'.

Next suppose H is of the type $\mathbf{E}_6$ or $\mathbf{E}_7$. As before let λ be the infinitesimal character corresponding to λ'. Then $\Theta(S_{\lambda'}) \neq 0$. In addition $\Theta(S_\lambda)$ is a finite length $(\mathfrak{g}', K')$-module with the unique irreducible quotient isomorphic to $S_{\lambda'}$.

In summary we establish the strong duality for spherical representations in the split $\mathbf{E}_6$ and $\mathbf{E}_7$ cases, but only one for the dual pair in the split $\mathbf{E}_8$ case.

Reference:

H.Y. Loke, G. Savin, "Duality for spherical representations in exceptional theta correspondences". Transactions of the American Mathematical Society, 371, No. 9 (2019): 6359-6375.