NATIONAL UNIVERSITY OF SINGAPORE

Qualifying Exam Paper I: Algebra

August 2021

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Get ready a signed copy of the **Exam declaration form** for this exam.
2. Use **A4** size paper and **pen** (blue or black ink) to write your answers.
3. Write down your student number clearly on the **top left** of every page of the answers.
4. Write on one side of the paper only. Write the question number and page number on the **top right** corner of each page (e.g. Q1P1, Q1P2, . . . , Q2P1, . . .).
5. This examination paper contains **SIX (6)** questions and comprises **THREE (3)** pages. Answer **ALL** questions.
6. The total mark for this paper is ONE HUNDRED (100).
7. This is an **OPEN BOOK** examination: you are allowed to use any book or lecture notes (hard copies or PDF), but you are not allowed to search online or discuss with others.
8. **Please start each question on a NEW page.**
9. You may use any calculator. However, you should lay out systematically the various steps in the calculations.
10. Join the **Zoom conference** and turn on the video setting at all time during the exam. Adjust your camera such that **your face and upper body including your hands** are captured on Zoom.
11. You may go for a short toilet break (not more than **5 minutes**) during the exam.

    At the end of the exam (12 noon):

12. Scan or take pictures of your work (make sure the images can be **read clearly**) together with the declaration form;
13. Merge all your images into **One (1) PDF** file (arrange them in the order: **Declaration form, Q1 to Q6** in their page sequence);
14. Name the PDF file by MatricNo_Module Name (e.g. **A1234567R_QE_Alg**);
15. Email your PDF to the examiner at: **MATZDQ@NUS.EDU.SG**, titled: **QE Algebra**;
16. Your PDF should reach examiner's email box at **12:15pm**. Exam answers that are not submitted by then will not be accepted, unless there is a valid reason.

Question 1 [18 marks]

Let G be a finite group.

(i) State the definition of a nilpotent group.

(ii) State the definition of a Sylow p-subgroup.

(iii) State the definition of the normalizer $N_G(H)$ of a subgroup $H \subseteq G$.

(iv) Let $H \lhd G$ be a normal subgroup and let P be a Sylow p-subgroup of H. Show that $G = H\, N_G(P)$ and $[G : H]$ divides $|N_G(P)|$.

(v) Show that if every maximal subgroup of G is normal then G is nilpotent.

Question 2 [16 marks]

(i) State the definition of a (left) Notherian ring.

(ii) Let A be a commutative ring with 1. Let $A[[x]] = \{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in A\ (\forall\, i)\}$ be the former power series ring. Show that if A is Noetherian then so is $A[[x]]$.

Question 3 [18 marks]

Let V be an n-dimensional vector space over a field K and let $T : V \to V$ be a K-linear transformation such that the characteristic polynomial of T splits as

$$P_T(x) = (x - \lambda_1) \cdots (x - \lambda_n)$$

with all $\lambda_i \in K$. Let $R := K[x]$. It is known that V is an R-module where the action $r = f(x)$ on $v \in V$ is given by: $rv = f(x)v := f(T)(v)$.

(i) State the definition of a matrix $A \in M_n(K)$ being a Jordan canonical form, and the definition of a basis B of V being a Jordan canonical basis giving rise to the Jordan canonical form A of T.

(ii) Show that V is a torsion R-module and $m(x)V = 0$ for some monic polynomial $m(x) \in R$. We fix such a $m(x)$ with $\deg m(x)$ being the smallest.

(iii) It is known that

$$(*) \quad V = R/(p_1^{s_1}) \oplus \cdots \oplus R/(p_m^{s_m})$$

(up to isomorphism), for some irreducible monic polynomials $p_i \in R$ and integers $s_i \geq 1$. Show that $p_i(x) = (x - \alpha_i)^{t_i}$ for some $\alpha_i \in \{\lambda_1, \ldots, \lambda_n\}$ and integer $t_i \geq 1$.

(iv) Use the decomposition $(*)$ above to determine the polynomial $m(x)$ in (ii).

(v) Use the decomposition $(*)$ above to determine/construct a Jordan canonical form $A \in M_n(F)$. Find a Jordan canonical basis B of V which gives rise to the Jordan canonical form A of T.

Question 4 [16 marks]

(i) State the definition of a local ring, and the definition of a (left) Noetherian ring.

In the following, R is Notherian local ring with maximal ideal M.

(ii) It is known that $F := R/M$ is a field and M/M^2 is naturally an F-module. Show that $d := \dim_F M/M^2 < \infty$.

(iii) Show that M can be generated by d elements (as an R-module), but M cannot be generated by $d - 1$ elements.

Question 5 [16 marks]

Let $n \geq 2$ and let F $(\subset \mathbb{C})$ be the splitting field of $x^n - 1$ over $\mathbb{Q}$.

(i) State the definition of a Galois extension.

(ii) Show that F is a simple extension of $\mathbb{Q}$, i.e., $F = \mathbb{Q}[\zeta_n]$ for some $\zeta_n \in F$.

(iii) Determine the dimension $[F : \mathbb{Q}]$.

(iv) Is F Galois over $\mathbb{Q}$? If so, determine the Galois group Gal(F/$\mathbb{Q}$).

Question 6 [16 marks]

Let p be a prime and G a finite group of order p^n for some $n \geq 1$. Let $K \subset F$ be a Galois extension with Gal(F/K) = G.

(i) What is the dimension $[F : K]$ of the field extension $K \subset F$?

(ii) Show that for every $1 \leq r \leq n$ there is an intermediate field $K \subseteq L_r \subseteq F$ such that L_r is Galois over K and the dimension $[L_r : K] = p^r$.

(iii) Is F Galois over L_r?

END OF PAPER