NATIONAL UNIVERSITY OF SINGAPORE

Qualifying Exam Paper I: Algebra

August 2022

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1 Write down your student number clearly on the top left of every page of the answers. Do not write your name.

2 Write on one side of the paper only. Write the question number and page number (if a single question takes more than one page) on the top right corner of each page (e.g. Q1, Q2, . . .).

3 This examination paper contains **SEVEN (7)** questions and comprises **TWO (2)** pages, including this one. Answer ALL questions.

4 The total mark for this paper is **ONE HUNDRED (100)**.

5 This is a **CLOSED BOOK** examination.

Convention: Unless specified otherwise, we assume: rings are with $1 \neq 0$; modules are left modules; ring homomorphisms preserve the multiplicative identity.

Q1 [15 marks] Let G be a group of order 12. Show that some Sylow subgroup of G is normal.

Q2 [15 marks] Let M be a Noetherian R-module for some ring R. Let $f : M \to M$ be a R-module homomorphism. Prove that f is surjective, then f is an isomorphism.

Q3 [10 marks] Let F be an algebraically closed field. Prove that F is an infinite set.

Q4 [15 marks] Let $GL_2(F_2)$ be the general linear group over the finite field F_2 of 2 elements. Determine the number of conjugacy classes of $GL_2(F_2)$.

Q5 [15 marks] Let M be an abelian group generated by x and y subject to the relations $7x + 5y = 0$ and $2x - 4y = 0$. Prove that $M \cong \mathbb{Z}/38\mathbb{Z}$.

Q6 [15 marks] Let R be a commutative ring with a unique maximal ideal M. Show that every element in $R - M$ is invertible. (Here $R - M = \{r \in R | r \notin M\}$.)

Q7 [15 marks] Let M, N, K be R-modules for some ring R. Then the sequence $M \to N \to K \to 0$ is exact if and only if the induced sequence $0 \to Hom_{R\text{-mod}}(K, D) \to Hom_{R\text{-mod}}(N, D) \to Hom_{R\text{-mod}}(M, D)$ is exact for all R-module D.

End of Paper