NATIONAL UNIVERSITY OF SINGAPORE

Qualifying Exam Paper I: Algebra

Aug 2023

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1 Write down your student number clearly on the top left of every page of the answers. Do not write your name.

2 Write on one side of the paper only. Write the question number and page number (if a single question takes more than one page) on the top right corner of each page (e.g. Q1, Q2, ...).

3 This examination paper contains **SEVEN (7)** questions and comprises **TWO (2)** pages, including this one. Answer ALL questions.

4 The total mark for this paper is **ONE HUNDRED (100)**.

5 This is a **CLOSED BOOK** examination.

Convention: Unless specified otherwise, we assume: rings are with $1 \neq 0$; modules are left modules; ring homomorphisms preserve the multiplicative identity.

Q1 [10 marks] Let G be group with a normal subgroup N. We denote by G/N the quotient group. Prove that if both N and G/N are solvable, then G is also solvable. (Remark: State the definition of solvable groups you used here first.)

Q2 [10 marks] Let G be a non-trivial finite group of order $n > 1$. Prove that any simple G-module over a field F has (F-)dimension $< n$.

Q3 [10 marks] Let I be an ideal of a commutative ring R. Let P be a projective R-module. Prove that the natural map $I \otimes_R P \to IP$, $r \otimes p \to rp$ is an isomorphism of R-modules.

Q4 [20 marks] Let p be an odd prime and n be a positive integer. Let $G = (\mathbb{Z}/p^n\mathbb{Z})^*$ be the (multiplicative) group of units in the ring $\mathbb{Z}/p^n\mathbb{Z}$.

(1) Prove that any Sylow p-subgroup in G is cyclic.
(2) Prove that G is cyclic. (Remark: You can freely use item (1) when proving item (2).)

Q5 [15 marks] Let M be an indecomposable, Neotherian and Artinian R-module for some ring R. Prove that any $f \in \mathrm{End}_R(M)$ is either an isomorphism or nilpotent.

Q6 [15 marks] Let F be a field. Prove that $A \in \mathrm{Mat}_{n\times n}(F)$ is similar to its transpose. (Remark: You will be awarded 10 marks if you prove the statement assuming F is algebraically closed.)

Q7 [20 marks] Determine whether the following statements are TRUE or FALSE. You do NOT need to justify your answer.

(1) Any finitely generated module over a PID is injective.
(2) An algebraically closed field can not be finite.
(3) Any symmetric matrix in $\mathrm{Mat}_{n\times n}(\mathbb{R})$ is diagonalizable.
(4) The polynomial ring $F[x_1, x_2, \ldots, x_n]$ is a unique factorization domain for any field F.

End of Paper