National University of Singapore

Qualifying Exam Paper I: Algebra

January 2022

Time allowed: 3 hours

INSTRUCTIONS TO CANDIDATES

1 Write down your student number clearly on the top left of every page of the answers. Do not write your name.

2 Write on one side of the paper only. Write the question number and page number (if a single question takes more than one page) on the top right corner of each page (e.g. Q1P1, Q1P2, . . . , Q2P1, . . .).

3 This examination paper contains **SEVEN (7)** questions and comprises **TWO (2)** pages, including this one. Answer ALL questions.

4 The total mark for this paper is **ONE HUNDRED (100)**.

5 This is a **CLOSED BOOK** examination.

Convention: Unless specified otherwise, we assume: rings are with $1 \neq 0$; modules are left modules; ring homomorphisms preserve the multiplicative identity.

Q1 [15 marks] Let G be a (possibly infinite) group whose automorphism group Aut(G) is cyclic. Prove that G is abelian.

Q2 [15 marks] Let G be a finite group of order p^n for some prime p. Let $\mathbb{F}_p$ be the finite field of p elements. Prove that any irreducible $\mathbb{F}_p[G]$-module trivial.

(Here a trivial $\mathbb{F}_p[G]$-module is a one dimensional $\mathbb{F}_p$-vector space where all elements of G acts as identity.)

Q3 [15 marks] Let R be a ring. Let $0 \to M \to Q \to L \to 0$ and $0 \to M \to Q' \to L' \to 0$ be short exact sequences of R-modules where Q and Q' are both injective. Prove that $Q \oplus L' \cong Q' \oplus L$.

(You can assume that pushout and pullback exist in the category of R-modules.)

Q4 [10 marks] Let M be an abelian group generated by x and y subject to the relations $x + 2y = 0$ and $4x = 0$. Prove that $M \cong \mathbb{Z}/8\mathbb{Z}$.

Q5 [15 marks] Let R be a PID. Let $\phi : R^n \to R^n$ be an R-module homomorphism for some $n \geq 1$.

(1) Prove that ϕ is surjective if and only if ϕ is an isomorphism.
(2) Give an example of $\phi : R^n \to R^n$ that is injective but not isomorphic.

Q6 [15 marks] Let R be a commutative ring. We denote by J the Jacobson radical of R, i.e., the intersection of all maximal ideals.

(1) An element x belongs to J if and only if $1 - rx$ is a unit for any $r \in R$.
(2) *(Nakayama's Lemma) If M is a finitely generated R-module such that $JM = M$, then $M = 0$.*

Q7 [15 marks] Let R be an Artinian ring. Prove that R has only finitely many maximal ideals.

End of Paper