NATIONAL UNIVERSITY OF SINGAPORE

Qualifying Exam Paper I: Algebra

Jan 2023

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1 Write down your student number clearly on the top left of every page of the answers. Do not write your name.

2 Write on one side of the paper only. Write the question number and page number (if a single question takes more than one page) on the top right corner of each page (e.g. Q1, Q2, . . .).

3 This examination paper contains **SEVEN (7)** questions and comprises **TWO (2)** pages, including this one. Answer ALL questions.

4 The total mark for this paper is **ONE HUNDRED (100)**.

5 This is a **CLOSED BOOK** examination.

Convention: Unless specified otherwise, we assume: rings are with $1 \neq 0$; modules are left modules; ring homomorphisms preserve the multiplicative identity.

Q1 [10 marks] Let k be a field. Prove that any Noetherian k-module is also Artinian.

Q2 [10 marks] Show by an example that $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module. (You need to state the definition of injective modules you are using. You need to show sufficient details.)

Q3 [15 marks] Let R be an integral domain with a prime ideal P. Let $D = \{r \in R | r \notin P\}$.

(1) Prove that D is a multiplicatively closed, that is, if $a, b \in D$, then $ab \in D$.
(2) Let $D^{-1}R$ be the localization of R with respect to D. Prove that $D^{-1}R$ is a local ring, that is, it has a unique maximal ideal.

Q4 [15 marks] Let $S = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) | A^2 = I\}$. Here I is the identity matrix. The general linear group $GL_2(\mathbb{C})$ acts on S via conjugation. Determine the number of orbits. Justify your results.

Q5 [15 marks] Let I be a left ideal of a ring R such that $I^n = (0)$ for some $n \geq 0$. Let $\phi : M \to N$ be a R-module homomorphism. Prove that if the induced map $\phi : M/IM \to N/IN$ is surjective, then ϕ is surjective.

Q6 [15 marks] Let G be a finite group and V be a finite-dimensional simple G-module over $\mathbb{C}$. It is known that there exists a non-degenerate G-invariant Hermitian form on V. Prove that such Hermitian form is unique up to scalar multiplications.

(Recall a G-invariant Hermitian form on V is a map $(\cdot,\cdot) : V \times V \to \mathbb{C}$ such that for $v, w \in V$, we have $(av, w) = (v, \overline{a}w) = a(v, w)$ for $a \in \mathbb{C}$; $(v, w) = (gv, gw)$ for $g \in G$. Here $\overline{a}$ denotes the complex conjugate of a.)

Q7 [20 marks] Determine whether the following statements are TRUE or FALSE. You do NOT need to justify your answer.

(1) Any finite integral domain is a field.
(2) Let F be a field. Then any subgroup $G \subset F^*$ of the multiplicative group is cyclic.
(3) Let R be a commutative Noetherian ring. Then the polynomial ring $R[x]$ is also Noetherian.
(4) Every unique factorization domain is a principle ideal domain.

End of Paper