Algebra Qualifying Exam
January 2024

January 1, 2024

Problems

1. Classify all groups of order 21.
2. Compute the Galois group of the polynomial $x^4 + 2$ over $\mathbf{Q}$, and draw the lattice of all subfields of its splitting field.
3. Compute the group $\mathrm{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/6\mathbf{Z}, \mathbf{Z}/8\mathbf{Z})$.
4. Precisely state the classification theorem for finitely generated modules over a PID.
5. Prove that if A is a Noetherian ring and M is a finitely generated A-module, then any A-submodule $N \subset M$ is finitely generated.
6. Give precise definitions or statements of the following notions:
   i. A unique factorization domain.
   ii. An integrally closed domain.
   iii. The Hilbert basis theorem.
7. Determine the number of irreducible representations of the group S_4, along with their dimensions.
8. Classify all abelian groups of order 360.
9. Give examples of each of the following:
   i. A commutative ring with exactly one maximal ideal, which is not a field.
   ii. An integral domain which is not a unique factorization domain.
   iii. A commutative ring with infinitely many elements but exactly 6 invertible elements.