NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER I 2021 – 2022

Ph.D. QUALIFYING EXAMINATION

PAPER 2

ANALYSIS

Time allowed: Three and a half $\left(3\,\frac{1}{2}\right)$ hours (including uploading)

INSTRUCTIONS TO CANDIDATES

1. Please follow the instructions in the document "**Instructions for the e-Examination of the Qualifying Examination, Paper II, Analysis**".

2. This examination paper contains a total of **THREE (3)** questions and comprises **SIX (6)** printed pages (including this front page).

3. Answer **ALL** questions.

4. Please start each question on a new page. You may answer subparts of the same question in one page.

5. This is an Open Book Exams., but internet searching is NOT ALLOWED. However, hard copies or PDFs of textbooks are ALLOWED.

6. Please scan or take a photo of each page of your answer sheets, together with the signed **Exam Declaration Form**. As far as possible combine all pages into one PDF file, and send to

   matlmc@nus.edu.sg

   The title of you e-mail should be

   QE Paper II Analysis

   You can only submit one time (no correction or resubmission). In case of multiple submissions, the earliest version would be treated as the final version, and it is the only version that would be graded. .

Answer **ALL** *the three questions.*

Question 1 [35 marks]

(i) Consider the mapping

$$\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$$

$$(x_1, x_2, x_3) \mapsto (a x_1 + b x_2, c x_2 + d x_3) \in \mathbb{R}^2 \quad \text{for} \quad (x_1, x_2, x_3) \in \mathbb{R}^3 .$$

Here a, b, c and d are fixed numbers, with

(1.1) $$a \neq 0 \quad \text{and} \quad c \neq 0 .$$

$(\mathbf{i})_a$ Compute the Jacobian matrix

$$\mathbf{J}\,\mathbf{f}(x_1, x_2, x_3) \quad \text{for} \quad (x_1, x_2, x_3) \in \mathbb{R}^3 .$$

[You may like to refer to (1.2) for a general expression of the Jacobian matrix.] Your answer (in its simplest form) should be in terms of a, b, c and d.

$(\mathbf{i})_b$ Show that the Jacobian matrix $\mathbf{J}\,\mathbf{f}(x_1, x_2, x_3)$ has rank two for all $(x_1, x_2, x_3) \in \mathbb{R}^3$.

$(\mathbf{i})_c$ Show that there exists a smooth mapping

$$\mathbf{s} : \mathbb{R}^2 \to \mathbb{R}^3$$

$$(y_1, y_2) \mapsto (s_1(y_1, y_2), s_2(y_1, y_2), s_3(y_1, y_2)) \in \mathbb{R}^3$$

$$\text{for} \quad (y_1, y_2) \in \mathbb{R}^2 ,$$

such that

$$s_3(y_1, y_2) \equiv 1 \quad \text{and} \quad \mathbf{f} \circ \mathbf{s}(y_1, y_2) = (y_1, y_2) \quad \text{for all} \quad (y_1, y_2) \in \mathbb{R}^2 .$$

Find

$$s_1(y_1, y_2) \quad \text{and} \quad s_2(y_1, y_2)$$

in terms of y_1, y_2, a, b, c and d. Your answers should be in their simplest forms. Justify your answers.

– *Question* 1 *continues on the next page.* –

Question 1 *continues...*

(ii) For integers M and n satisfying $M > n \geq 2$, consider a smooth mapping

$$\mathbf{F} : \mathbb{R}^M \to \mathbb{R}^n$$

$$\mathbf{x} \mapsto (F_1(\mathbf{x}), \cdots, F_n(\mathbf{x})) \in \mathbb{R}^n \qquad \text{for } \mathbf{x} = (x_1, \cdots, x_M) \in \mathbb{R}^M,$$

with

$$\mathbf{F}(0, \cdots, 0) = (0, \cdots, 0).$$

Here $F_1, \cdots, F_n$ are smooth functions on $\mathbb{R}^M$. The Jacobian matrix is given by:

$$(1.2) \qquad \mathbf{J}\,\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{x}) \cdots & \frac{\partial F_1}{\partial x_M}(\mathbf{x}) \\ \frac{\partial F_2}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial F_2}{\partial x_n}(\mathbf{x}) \cdots & \frac{\partial F_2}{\partial x_M}(\mathbf{x}) \\ \cdot & & & \\ \cdot & & & \\ \frac{\partial F_n}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial F_n}{\partial x_n}(\mathbf{x}) \cdots & \frac{\partial F_n}{\partial x_M}(\mathbf{x}) \end{pmatrix} \qquad \text{for} \quad \mathbf{x} \in \mathbb{R}^M.$$

Assume that

$$\operatorname{Det} \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{0}) & \cdots & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{0}) \\ \frac{\partial F_2}{\partial x_1}(\mathbf{0}) & \cdots & \cdots & \frac{\partial F_2}{\partial x_n}(\mathbf{0}) \\ \cdot & & & \\ \cdot & & & \\ \frac{\partial F_n}{\partial x_1}(\mathbf{0}) & \cdots & \cdots & \frac{\partial F_n}{\partial x_n}(\mathbf{0}) \end{pmatrix} \neq 0.$$

Here

$$\mathbf{0} = (0, \cdots, 0) \in \mathbb{R}^M.$$

– *Question* 1 *continues on the next page*. –

Question 1 *continues...*

Define

$$\tilde{\mathbf{F}} : \mathbb{R}^M \to \mathbb{R}^M$$

$$\mathbf{x} \mapsto (F_1(\mathbf{x}), \cdots, F_n(\mathbf{x}), x_{n+1}, \cdots, x_M) \in \mathbb{R}^M$$

$$\text{for } \mathbf{x} = (x_1, \cdots, x_M) \in \mathbb{R}^M .$$

$(\mathbf{ii})_a$ Show that the Jacobian matrix $\mathbf{J}\tilde{\mathbf{F}}(\mathbf{0})$ is non-singular.

$(\mathbf{ii})_b$ Using the Inverse Function Theorem (you are not required to proof it), show that there exist an open set $\mathcal{O}$ in $\mathbb{R}^n$, which contains the origin (of $\mathbb{R}^n$), and a smooth mapping

$$\mathbf{S} : \mathcal{O} \to \mathbb{R}^M$$

$$(y_1, \cdots, y_n) \mapsto (S_1(y_1, \cdots, y_n), \cdots, S_M(y_1, \cdots, y_n)) \in \mathbb{R}^M$$

$$\text{for } (y_1, \cdots, y_n) \in \mathcal{O} \subset \mathbb{R}^n ,$$

so that

$$\mathbf{F} \circ \mathbf{S}(y_1, \cdots, y_n) = (y_1, \cdots, y_n) \qquad \text{for all } (y_1, \cdots, y_n) \in \mathcal{O} .$$

Note that if you use another method (not via the Inverse Function Theorem), you may not be awarded full credit for this part.

– Question 2 starts on the next page. –

Question 2 [35 marks]

Consider $L^2(\mathbb{R})$ – the collection of Lebesque measurable functions

$$f : \mathbb{R} \to [-\infty, +\infty]$$

such that

$$(\|f\|_2 :=) \left(\int_{\mathbb{R}} |f|^2 \, d\lambda \right)^{\frac{1}{2}} < \infty .$$

In the above, the integration is with respect to the Lebesque measure on the real line $\mathbb{R}$. Let

$$\mathbf{T} \subset L^2(\mathbb{R})$$

satisfy the following property (2.1).

(2.1) For any sequence

$$\{ f_i \} \subset \mathbf{T},$$

there exists a subsequence $\{ f_{i_j} \}$ (of $\{f_i\}$) and $f_\infty \in \mathbf{T}$ so that

$$\| f_{i_j} - f_\infty \|_2 \to 0 \quad \text{as} \quad j \to \infty .$$

(**i**) Is it true that $\mathbf{T}$ is bounded? That is to say, "Is it true that there is a (fixed) positive number C so that

$$\|f\|_2 \le C \quad \text{for all} \quad f \in \mathbf{T}\text{?"}$$

Justify your answer.

(**ii**) Does the following property (2.2) hold for $\mathbf{T}$?

(2.2) For any (small) number $\varepsilon > 0$, there exists a (large) number $R_\varepsilon > 0$ so that

$$\int_{\{ |x| \ge R_\varepsilon \}} |f|^2 \, d\lambda \le \varepsilon \quad \text{for all} \quad f \in \mathbf{T}.$$

Justify your answer.

(**iii**) Does the following property (2.3) hold for $\mathbf{T}$?

(2.3) For any (small) number $\tilde{\varepsilon} > 0$, there exists a (small) number $\rho_{\tilde{\varepsilon}} > 0$ so that for any (fixed) $y \in \mathbb{R}$ with $|y| \le \rho_{\tilde{\varepsilon}}$, we have

$$\int_{\mathbb{R}} |f(x+y) - f(x)|^2 \, d\lambda_x \le \tilde{\varepsilon} \quad \text{for all} \quad f \in \mathbf{T} .$$

Justify your answer.

You are required to justify any non-standard result that you use in answering this question. The Monotone Convergence Theorem, Fatou's Lemma and the Lebesque Dominated Convergence Theorem are treated as standard results.

– *Question* 3 *starts on the next page*. –

Question 3 [30 marks]

(a) Picard's Little Theorem (you are not required to prove it) states that for any *non-constant* entire function

$$F : \mathbb{C} \to \mathbb{C},$$

$\mathbb{C} \setminus F(\mathbb{C})$ contains at most one point. Let

$$f : \mathbb{C} \to \mathbb{C},$$

be an *non-constant* entire function, satisfying

$$(3.1) \qquad f(1 - z) = 1 - f(z) \quad \text{for all} \quad z \in \mathbb{C}.$$

$(\mathbf{a})_i$ Find an example of an entire function F so that $\mathbb{C} \setminus F(\mathbb{C})$ contains exactly one point.

$(\mathbf{a})_{ii}$ Find an example of a non-constant entire function f satisfying (3.1).

$(\mathbf{a})_{iii}$ Is it always true that $f(\mathbb{C}) = \mathbb{C}$? Justify your answer. You are allowed to apply Picard's Little Theorem (without proving it).

(b) Let

$$I(x) := \int_{-\infty}^{\infty} \frac{dy}{(1 + y^2)[1 + (x - y)^2]} \quad \text{for} \quad x \in \mathbb{R}.$$

$(\mathbf{b})_i$ Is it true that

$$I(x) = I(-x) \quad \text{for all} \quad x \in \mathbb{R}?$$

Justify your answer.

$(\mathbf{b})_{ii}$ Using the Cauchy Integral Formula, evaluate $I(x)$ for all $x \in \mathbb{R}$. Your answer (in its simplest form) should be in terms of x and other constant(s). Justify your answer.

{ Note that if you use another method [not the Cauchy Integral Formula], you may not be awarded full credit for this part. }

– **End of the Examination Paper.** –