Analysis 2022 August

(1) Let $f \in L_{loc}(\mathbb{R}^n)$ and define [5 marks]

$$Mf(x) = \sup_{r>0} \int_{B_r(x)} |f(y)|dy/|B_r(x)| \text{ where } B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}.$$

Show that maximal function Mf is Borel measurable.

(2) (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Show that for given any $x_0 \in \mathbb{R}$, there exists $l \in \mathbb{R}$ such that $f(x) \geq f(x_0) + l(x - x_0)$ for all $x \in \mathbb{R}$.

(b) Use part (a) to prove Jensen's inequality: (let $g : [a,b] \to \mathbb{R}$ be continuous and $\alpha : [a,b] \to [0,1]$ be non-decreasing such that $\alpha(a) = 0, \alpha(b) = 1$)

$$f\left(\int_a^b g(x)d\alpha(x)\right) \leq \int_a^b f(g(x))d\alpha(x). \text{ (Riemann Stieltjes integral)}$$

(c) Suppose $h : [0,\infty) \to [0,\infty)$ is continuous, $1 \leq p < \infty, r > 0$. Use part (b) or otherwise to show that

$$\left(\int_0^x h(t)dt\right)^p \leq (p/r)^{p-1} x^{r(1-\frac{1}{p})} \int_0^x h(t)^p t^{p-r-1+\frac{r}{p}} dt \text{ for } x > 0.$$

(d) With the help of part (c), show that

$$\int_0^\infty \left(\int_0^x h(t)dt\right)^p x^{-r-1}dx \leq (p/r)^p \int_0^\infty h(t)^p t^{p-r-1}dt.$$

(The above is known as Hardy's inequality.) [20 marks]

(3) Let Ω be a bounded open set in $\mathbb{R}^n$ and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that the function $f(\cdot, t)$ is measurable for each $t \in \mathbb{R}$, and $f(x, \cdot)$ is continuous for almost all $x \in \Omega$. Show that $f(x, u(x))$ is measurable if u is a measurable function on Ω. If the function f is bounded from below and $u_k \to u$ in $L^1(\Omega)$, show that [8 marks]

$$\int_\Omega f(x, u(x))dx \leq \liminf_{k\to\infty} \int_\Omega f(x, u_k(x))dx.$$

(4) Let $\{f_n\} \subset L^p$ such that $\|f_n - f\|_{L^p} \to 0$, $1 \le p < \infty$. Show that [10 marks]

(a) $f_n \to f$ in measure;

(b) $\int_{E_k} |f_n|^p dx \to 0$ uniformly in n for any sequence of measurable sets E_k with $|E_k| \to 0$;

(c) for all $\varepsilon > 0$, there exists E_ε such that $|E_\varepsilon| < \infty$ and

$$\int_{\mathbb{R}^n \setminus E_\varepsilon} |f_n|^p dx < \varepsilon \quad \text{for all } n.$$

(5) (a) Compute $\displaystyle\int_0^\infty \frac{\log(1+x^2)}{1+x^2} dx$.

(b) A family $\mathcal{F}$ of functions on an open set Ω is said to be locally uniformly bounded if given any $x \in \Omega$, there exist $r_x, M > 0$ such that $|f(y)| \le M$ for all $y \in B_{r_x}(x) = \{|y-x| < r_x\} \subset \Omega$ and $f \in \mathcal{F}$. Show that any locally uniformly bounded family of analytic functions on an open connected set $\Omega \subset \mathbb{C}$ is equi-continuous on any compact subset of Ω. [18 marks]

(6) Let $f : (-1,1) \times (-1,1) \to \mathbb{R}$ be such that both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and bounded, show that f is continuous. [5 marks]

(7) Let Ω be a bounded open set in $\mathbb{R}^n$. Show that there exist countable disjoint closed cubes $\{Q_i\}$ in Ω such that $|\Omega \setminus (\bigcup_i Q_i)| = 0$. [5 marks]

(8) Answer only true or false for each of the following statements. 0.5 marks will be deducted for each wrong answer. [9 marks].

(a) Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be such that $f(x,\cdot)$ and $f(\cdot,x)$ are continuous function on $[0,1]$. If $f(x,y) = f(y,x)$ for all $x, y \in [0,1]$, then f is continuous on $[0,1] \times [0,1]$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : (-1,1) \to \mathbb{R}$ (both infinitely differentiable) such that $g(0) = 0$ and

$$\limsup(|f^{(n)}(0)|/n!)^{1/n} = 0 \text{ and } \limsup(|g^{(n)}(0)|/n!)^{1/n} = 1.$$

Then $f \circ g$ is real analytic on $(-1,1)$.

(c) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. If there are $z_1 \ne z_2$ such that $f(z_1) = f(z_2)$, then there exists z_0 on the line segment connecting z_1 to z_2 such that $f'(z_0) = 0$.

(d) If $f : \Omega \to \mathbb{R}$ is measurable where Ω is a bounded open set in $\mathbb{R}^n$, then for any $\varepsilon > 0$, there exists a compact set $K \subset \Omega$ and $M \in \mathbb{N}$, such that $|f(x)| \le M$ for all $x \in K$, $|\Omega \setminus K| < \varepsilon$.

(e) If f is an analytic function on $B_r(1+i) = \{z \in \mathbb{C} : |z - 1 - i| < r\}$, then $f(z) = \sum_{k=0}^{\infty} a_k(z - (1+i))^k$ on $B_r(1+i)$.

(f) If f is continuous and of bounded variation on $[a, b]$, then f is absolutely continuous on $[a, b]$.

(9) For at most **Four** (4) of the following statements, prove or disprove each of the statement considered. [20 marks]

(a) Let $\{f_n\} \subset \mathcal{R}(I)$ (Riemann integrable on the interval I) and $f_n(x) \to f(x)$ for all x. If $f \in \mathcal{R}(I)$ and $\{f_n\}$ is uniformly bounded, then (Riemann integral) $\int_I f dx = \lim_{n\to\infty} \int_I f_n dx$.

(b) Let f be a infinitely differentiable on $\mathbb{R}^n$ such that $f(x) \ge 0$, $f = 0$ outside $B_r(0)$, then $\frac{\partial f}{\partial \nu} = 0$ on $\partial B_r(0)$ where $\nu(x)$ be the outward normal vector. Then f must be a zero function.

(c) If f is Lebesgue measurable, then there exists Borel function g such that $f = g$ a.e..

(d) Let (E_i) be a sequence of measurable sets in $\mathbb{R}$ such that $\sum_{i=1}^{\infty} |E_i| < \infty$. Let $F = \cap_{k=1}^{\infty} \cup_{i=k}^{\infty} E_i$. Then $|F| = 0$.

(e) Let $\{f_n\}$ be a sequence of functions of bounded variation on $[a, b]$. Let $f(x) = \lim_{n\to\infty} f_n(x)$ (limit exist). Suppose there exists $M > 0$ such that

$$\sum_{i=1}^{N} |f_n(x_i) - f_n(x_{i-1})| \le M \quad \text{for all } n$$

for any partition $x_0 = a < x_1 < \cdots < x_N = b$.

Then f is also a function of bounded variation on $[a, b]$.

(f) Let Ω be a bounded open connected set in $\mathbb{R}^N$ and $f : \overline{\Omega} \subset \mathbb{R}^N \to \mathbb{R}^N$ such that f is continuous on $\overline{\Omega}$ and continuously differentiable on Ω. Let

$$f^{-1}(0) = \{x \in \Omega : \text{ such that } f(x) = 0\}.$$

Suppose f is none zero on the boundary of Ω. Then $f^{-1}(0)$ must be finite if $|J_f(y)| \ne 0$ for all $y \in f^{-1}(0)$ where J_f is the Jacobian of f.