Ph.D. Qualifying Examination 2023 August (Analysis)

Do all the questions.

Question 1 [14 marks]

Answer only true or false for each of the following statements. 1 mark will be deducted for each wrong answer.

(a) The product of functions of bounded variations is also a function of bounded variation.

(b) For any $1 \leq p < \infty$, convolutions of L^p functions is measurable.

(c) The uniform limit of a sequence of functions of bounded variations on $[a, b]$ is also a function of bounded variation.

(d) Any open set in $\mathbb{R}^n$ is the union of at most countable open sets that are pairwise disjoint and connected.

(e) If f is nonconstant analytic on Ω (domain), then, for all $B_r(a) \subset \Omega$, there exists $b \in B_r(a)$ such that $|f(b)| > |f(a)|$.

(f) Let f be an entire function on $\mathbb{C}$. If there exist uncountable number of points such that f has the same value at those points, then f is a constant.

(g) Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Let $\{\phi_k\}$ be a sequence of continuous functions such that it converges uniformly on any compact subset of $\mathbb{R}^n$ and $\phi_k \to f$ in $L^p(\mathbb{R}^n)$. If $\phi_k(x_0) \to f(x_0)$ with $x_0 \in \mathbb{R}^n$, then x_0 is a Lebesgue's point of f.

Question 2 [10 marks]

Let $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a measurable function such that both

$$\mathbf{esssup}_{y\in\mathbb{R}^n} \int |K(x,y)|dx, \mathbf{esssup}_{x\in\mathbb{R}^n} \int |K(x,y)|dy \text{ are finite.}$$

Let $f \in L^2(\mathbb{R}^n)$ and define $h(f)(x) = \int K(x,y)f(y)dy$. Show that $h(f) \in L^2(\mathbb{R}^n)$.

Question 3 [10 marks]

Let g be a given continuous function on the circle $\{z \in \mathbb{C} : |z| = 1\}$. Show that the function $f(z) = \int_{|\xi|=1} \frac{g(\xi)}{\xi - z} d\xi$ is analytic for all z with $|z| \neq 1$.

Question 4 [8 marks]

Let f, g be Riemann integrable on $[0,1]$. If g is a periodic function on $\mathbb{R}$ with period 1, show that

$$\lim_{k\to\infty}\int_0^1 f(x)g(kx)dx = \lim_{k\to\infty}\int_0^1 f(x)g(kx+c)dx$$

for any constant $c \in \mathbb{R}$ (you may assume all limits exist).

Question 5 [8 marks]

$f : \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous (LSC) if (and only if) $\liminf_{x\to x_0} f(x) \geq f(x_0)$ for all $x_0 \in \mathbb{R}^n$, show that

(i) $\{f(x) > t\}$ is open for all t;

(ii) f has a minimum on any compact subset in $\mathbb{R}^n$;

(iii) $\sup_n f_n$ is LSC if f_n are LSC for all n.

Question 6 [10 marks]

Let $\Phi : [0,\infty) \to [0,\infty)$ be a continuous non-decreasing concave function with $\Phi(0) = 0$ and let $f_n \to f$ a.e. on $\mathbb{R}^n$. If $\int_{\mathbb{R}^n} \Phi(|f_n|)dx \leq M < \infty$ for all $n \in \mathbb{N}$, show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^n} \Phi(|f_n - f|) - \Phi(|f_n|) + \Phi(|f|)dx = 0.$$

Question 7 [8 marks]

Let D be the open unit disk in $\mathbb{C}$ and $f : D \to \mathbb{C}$ be analytic such that $|f(z)| \leq |z|/2$ for all z, show that

$$\frac{|f'(z)|}{1 - 4|f(z)|^2} \leq \frac{1}{2(1-|z|^2)} \quad \text{for all } z \in D.$$

Question 8 [7 marks]

Let $f : E \to \mathbb{R}$ be a measurable function with $|E| < \infty$. Show that given any $\varepsilon > 0$, there exist $M > 0$ and a compact set $K \subset E$ such that $|E \setminus K| < \varepsilon$ and $e^{f(x)} \leq M$ for all $x \in K$.

Question 9 [25 marks]

For at most **five** (5) of the following statements, prove or disprove each of the statement considered.

(a) Let f be a continuous function on $[a,b]$ and $\phi,\psi:[a,b]\to\mathbb{R}$ be monotone functions. If $\phi=\psi$ except countable number of points, then

$$\int_a^b f d\phi = \int_a^b f d\psi.$$

(b) If $\{f_n\}$ is a nondecreasing sequence of Riemann integrable functions on $R=[0,1]\times[0,1]$, that converges to 0, then $\lim_{n\to\infty}\int_R f_n dx = 0$.

(c) Let U be an open set in $\mathbb{R}^n$ and $f:U\to\mathbb{R}^n$ be a differentiable function (transformation). Then f maps measurable sets in U to measurable sets.

(d) Let $\{f_n\}$ be a sequence of harmonic functions on the open unit disk $\mathcal{D}$ that converges uniformly on any compact subset of $\mathcal{D}$. Then its limit f is also harmonic on the open unit disk.

(e) Let $\{f_n\}$ be a sequence of functions in $L^2(\mathbb{R}^n)$. If $\int f_n g dx \to 0$ for all $g\in L^2(\mathbb{R}^n)$, then $\int |f_n|^2 dx \to 0$.

(f) If f_n converges to f a.e., then it also converges to f in measure.

(g) Let Ω be a bounded open set in $\mathbb{R}^n$. Then the completion of

$$C_c(\Omega)=\{f:\Omega\to\mathbb{R}|\ f \text{ is continuous with compact support in } \Omega\}$$

is (under sup norm)

$$C_0(\Omega)=\{f:\Omega\to\mathbb{R}|\ f \text{ is continuous such that for all } \varepsilon>0,$$

$$\text{there exists compact set } K\subset\Omega \text{ with } |f(x)|<\varepsilon \text{ for } x\notin K\}.$$