Ph.D. Qualifying Examination January 2022 (Analysis)

Answer all the following.

(1) Let $f : (a, b) \to \mathbb{R}$ be such that for any $a < x < y < z < b$, we have

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x}.$$

Show that f is both left and right differentiable on (a, b). Show that both left and right derivatives are nondecreasing and continuous except at countably many points. Hence show that f is differentiable on (a, b) except at countably many points. [12]

(2) Let $\mathcal{D} \subset \mathbb{C}$ be the open unit disk (with center 0) and $\mathcal{H}$ be the family of analytic functions on $\mathcal{D}$. Let

$$A^2(\mathcal{D}) = \{f \in \mathcal{H} : \int_{\mathcal{D}} |f|^2 d\lambda < \infty\} \text{ where } \lambda \text{ is the Lebesgue measure on the complex plane } \mathbb{R}^2.$$

Show that $A^2(\mathcal{D})$ is a Hilbert space. [14]

Hint: first show that for any compact subset K of $\mathcal{D}$, there exists a constant C depending only on K such that

$$|f(z)| \leq C\|f\|_{L^2(\mathcal{D})} \text{ for all } z \in K.$$

(3) Let $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ be two bounded sequences. Show that [5]

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty}(a_n + b_n).$$

(4) Let w be a nonnegative measurable function on $\mathbb{R}^n$ and define $I_\alpha(x) = |x|^{\alpha-n}$ $(0 < \alpha < n)$. Show that there exist constants $a, b \in \mathbb{R}$ such that [7]

$$I_\alpha * w(x) = \int_0^\infty \int_{B(x,r^{1/(\alpha-n)})} w(y)dydr = a \int_0^\infty t^b \int_{B(x,t)} w(y)dydt.$$

(5) Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be absolutely continuous. Let $f^+(x) = \max\{0, f(x)\}$. Show that f^+ is differentiable a.e. and $(f^+)' = f'\chi_{\{f>0\}}$ a.e. . [8]

(6) Let $\mathcal{D}_r$ be the open disk in the complex plane with center 0 and radius r. For each $k \in \mathbb{Z}$, compute

$$\int_{\mathcal{D}_{2r}\setminus\mathcal{D}_r} z^k d\lambda \quad \text{and} \quad \int_{\mathcal{D}_{2r}\setminus\mathcal{D}_r} |z|^k d\lambda$$

where λ is the Lebesgue measure on $\mathbb{R}^2$. [8]

(7) Let Ω be a measurable set in $\mathbb{R}^d$ and $f_n, g_n : \Omega \to \mathbb{R}$ be two sequences of integrable functions on Ω. Suppose $f_n \geq g_n$ a.e. and $f_n \to f$, $g_n \to g$ in measure on Ω. Suppose

$$\lim_{n\to\infty} \int_\Omega g_n dx = \int_\Omega g dx.$$

Show that

$$\int_\Omega f dx \leq \liminf_{n\to\infty} \int_\Omega f_n dx.$$

[8]

(8) Let Ω be an open connected set in $\mathbb{R}^d$ and $\{u_n\} \subset C_c^1(\Omega)$ be such that both u_n and $\frac{\partial u_n}{\partial x_j}$ are Cauchy sequences in $L^2(\Omega)$ (where $1 \leq j \leq d$). Suppose u, v are L^2 limit of u_n and $\frac{\partial u_n}{\partial x_j}$ respectively. If $\phi \in C_c^1(\Omega)$ show that

$$\int_\Omega u \frac{\partial \phi}{\partial x_j} dx = -\int_\Omega v\phi dx.$$

[8]

(9) For at most **Six** (6) of the following statements, prove or disprove each of the statement below. [30].

(a) If a function f is Lipschitz continuous on $[a, b]$, $-\infty < a < b < \infty$, then there exist nondecreasing absolutely continuous functions f_1, f_2 on $[a, b]$ such that $f = f_1 - f_2$ on $[a, b]$.

(b) If $\{f_n\}$ is a nondecreasing sequence of Riemann integrable functions on $R = [0, 1] \times [0, 1]$ that converges to 0, then $\lim_{n\to\infty} \int_R f_n dx = 0$.

(c) A complex valued function is entire if and only if it is equal to a power series $\sum_{k=0}^{\infty} a_k z^k$ that converges everywhere.

(d) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable and $1 - 1$ and $T(U)$ is open whenever U is open, then T maps measurable sets to measurable sets.

(e) If f is an analytic function on an open connected set $\mathcal{D}$ in the complex plane, then it is either a constant function or it will map open subsets of $\mathcal{D}$ to open sets.

(f) If f and g are both real analytic functions at $x_0 \in \mathbb{R}$, then its product function fg is also real analytic at x_0.

(g) Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that $\liminf_{x \to x_0} f(x) \geq f(x_0)$ for all $x_0 \in \mathbb{R}^n$. Then f is Borel measurable.

(h) Let Ω be an open connected set in $\mathbb{R}^n$ and

$$C_0(\Omega) = \{f \in C(\Omega) : \forall \varepsilon > 0, \exists \text{ compact set } K \subset \Omega \text{ such that } |f(x)| < \varepsilon \ \forall x \notin K\}$$

where $C(\Omega)$ is the space of continuous functions on Ω. Then $C_0(\Omega)$ is dense in $L^2(\Omega)$.

(i) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuously differentiable function such that the Jacobian/derivative Df has nonzero determinant at the origin 0. Then there exists $\varepsilon > 0$ such that for all $y \in \mathbb{R}^3$ with $|y - f(0)| < \varepsilon$, the equation $f(x) = y$ has at least one solution.