Ph.D. Qualifying Examination 2023 January (Analysis)

Answer all the following.

Question 1 [12 marks]

Answer only true or false for each of the following statements. 1 mark will be deducted for each wrong answer.

(a) If a function f is integrable on $\mathbb{R}$, then

$$\int_a^b f(x)(\sin x)^k dx \to 0 \text{ as } k \to \infty.$$

(b) If $(x_0, y_0) \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{x \to x_0} f(x,y) = g(y) \text{ for all } y \text{ and } \lim_{y \to y_0} g(y) = l$$

then $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = l$.

(c) Let $f, g : \mathbb{R} \to \mathbb{R}$ be real analytic at x_0. Then their product $f(x)g(x)$ is also analytic at x_0.

(d) Let $\{a_n\}$ be a seqence of nonzero real numbers. Then

$$\lim_{n\to\infty} |a_{n+1}/a_n| = r \quad \text{implies } \limsup_{n\to\infty} |a_n|^{1/n} = r.$$

(e) Let $\{a_k\}$ be a sequence in $\mathbb{R}$ such that $\sum_{j=1}^{\infty} a_{\sigma(j)}$ for all permutations σ on $\mathbb{N}$. Then the series converges absolutely.

(f) Let μ, ν be Borel measures on $\mathbb{R}^n$ such that $\nu(E) = 0$ whenever $\mu(E) = 0$ $E \subset \mathbb{R}^n$. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\nu(E) < \varepsilon$ whenever $\mu(E) < \delta$.

Question 2 [8 marks]

Show that the function $f(z) = \sum_{n=1}^{\infty} 1/n^z$ is analytic on $A = \{z \in \mathbb{C} : Rez > 1\}$.

Question 3 [8 marks]

Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and $\mathcal{D} = \{(x, y) : x^2 + y^2 \leq 1\}$. Show that the function $F(x, y) = f(x + yi)1_{\mathcal{D}}(x, y)$ is Lebesgue measurable on $\mathbb{R}^2$ and

$$f(0) = \int_{\mathbb{R}^2} F(x, y) dxdy/\pi.$$

Question 4 [10 marks]

Let f_n and w be measurable functions on $\mathbb{R}^N$ such that $\int |f_n(x)|^p |w(x)| dx \leq M$ for all n where $0 < p \leq 1$. Suppose $f_n \to f$ a.e.. Show that

$$\int (|f_n(x)|^p - |f(x)|^p - |f_n(x) - f(x)|^p)|w(x)|dx \to 0 \quad \text{as } n \to \infty.$$

Question 5 [8 marks]

Let f be analytic on an open set $U \subset \mathbb{C}$ and $z_0 \in U$ with $f'(z_0) \neq 0$. Show that if γ is a sufficiently small circle centered at z_0 in anti-clockwise direction, then

$$\frac{2\pi i}{f'(z_0)} = \int_\gamma \frac{1}{f(z) - f(z_0)} dz.$$

Question 6 [18 marks]

Let $f : \Omega \to \mathbb{R}^n$ be such that f is continuously differentiable where Ω is a bounded open connected set in $\mathbb{R}^n$. For each $t \in \mathbb{R}$ define $f_t(x) = f(x) + tx$. Let $\mathcal{D}$ be any open connected set in Ω such that $\overline{\mathcal{D}} \subset \Omega$. Show that f_t is injective on $\mathcal{D}$ for sufficiently large t and $f(\partial\mathcal{D}) = \partial f(\mathcal{D})$.

Next, suppose $g : \Omega \to \mathbb{R}^n$ that is continuously differentiable with $g(x) = f(x)$ for $x \in \partial\mathcal{D}$. Show that

$$\int_{\mathcal{D}} |J_g| dx = \int_{\mathcal{D}} |J_f| dx \quad \text{where } J_g, J_f \text{ are Jacobians of } f \text{ and } g \text{ respectively.}$$

Question 7 [16 marks]

Let $\{a_k\}$ be a sequence of nonnegative real numbers and $1 \leq p < \infty$. Show that there exists a constant $C > 0$ independent of the above sequence such that

$$\| \sum_{k=1}^{\infty} a_k 1_{\tilde{B}_k} \|_{L^p} \leq C \| \sum_{k=1}^{\infty} a_k 1_{B_k} \|_{L^p}$$

where B_k and $\tilde{B}_k$ are balls for all k such that $\tilde{B}_k \subset 2B_k$ (the ball with the same center as B_k but with twice its radius).

Hint: use the converse of Hölder's inequality and maximal function, you may use the fact that if h^* is a maximal function of h, then

$$\|h^*\|_{L^p} \leq C\|h\|_{L^p} \quad \text{for } p > 1 \ .$$

Question 8 [20 marks] For at most **Four** (4) of the following statements, prove or disprove each of the statement considered. [20]

(1) Let a_n, r_n be a nonnegative sequence that converges to $a > 0$ and $r > 0$ respectively. Then $\lim_{n\to\infty} a_n^{r_n} = a^r$.

(2) Let f be continuous function on $[a, b]$ such that it is of bounded variation on $[a, b]$. If f is absolutely continuous on all $[c, d]$ with $[c, d] \subset (a, b)$, then f is absolutely continuous on $[a, b]$.

(3) Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence and $\{b_n\}_{n\in\mathbb{N}}$ be a convergence sequence that converges to b. Then

$$\liminf_{n\to\infty} a_n b_n = b \liminf_{n\to\infty} a_n.$$

(4) Let $f \in C^\infty[a, b]$ and $g \in BV[a, b]$. Then $fg \in BV$ and $(fg)' = f'g + fg'$.

(5) Uniform limit of a sequence of absolutely continuous functions on an interval $[a, b]$ is also absolutely continuous.

(6) Let f be an integrable function on an open interval (a, b) such that

$$\int_a^b f\phi' dx = 0 \quad \text{for all } \phi \in C_c^\infty(a, b)$$

the space of infinitely differentiable function with compact support in (a, b). Then f is a constant.