Ph.D. Qualifying Examination 2024 January (Analysis)

A closed book paper with helpsheet. Do all the questions. Theorems/results used should be clearly stated

Question 1 [**12 marks**]

Answer only true or false for each of the following statements.

(a) The product of absolutely continuous functions on $[a, b]$ is also absolutely continuous.

(b) For any $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n)$, $1 < p, q < \infty$ such that $1/p + 1/q = 1$, the convolution of f and g is continuous.

(c) The uniform limit of absolutely continuous functions on $[a, b]$ is also an absolutely continuous function.

(d) Any open set in $\mathbb{R}^n$ is the union of at most countable union of open cubes that are pairwise disjoint.

(e) The function $\sum_{k \in \mathbb{N}} e^{-kz}$ is analytic on $\{z \in \mathbb{C} : Re z > 0\}$.

(f) Limit (pointwise) of a sequence of convex functions on an interval I is also convex (assuming the limit function is finite everywhere).

(g) Let f, g, h be measurable functions on $\mathbb{R}^n$ and $1 < p, q < \infty$. If $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int |fgh| dx \leq (\int |f|^p |h| dx)^{1/p} (\int |g|^q |h| dx)^{1/q}.$$

(h) Let f be a continuous function on $[a, b]$ such that it is of bounded variation on $[a, b]$. If f is absolutely continuous on all $[c, d]$ with $[c, d] \subset (a, b)$, then f is absolutely continuous on $[a, b]$.

Question 2 [**8 marks**]

Let K be a compact metric space and $f_n : K \to \mathbb{R}$ be an increasing sequence (i.e., $f_{n+1}(x) \geq f_n(x)$ for all n and x) of continuous functions that converges to a function f everywhere on K. If f is also continuous on $[a, b]$, show that f_n converges to f uniformly.

Question 3 [6 marks]

Let f be a differentiable function on $[a, b]$ such that its derivative never vanishes. Show that $f'(x)f'(y) > 0$ for all $x, y \in [a, b]$.

Question 4 [10 marks]

Let D be the open unit disk in $\mathbb{C}$ and $f : D \to \mathbb{C}$ be analytic. If there exists $z_0 \in \mathbb{C}$ such that $|f(z) - z_0| \leq |z|/2$ for all z, show that

$$\frac{|f'(z)|}{1 - 4|f(z) - f(0)|^2} \leq \frac{1}{2(1 - |z|^2)} \quad \text{for all } z \in D.$$

Question 5 [12 marks]

Let f, g be locally integrable functions on $\mathbb{R}^n$ such that

$$\inf_{a \in \mathbb{R}} \int_B |f(x) - a| dx \leq 2 \int_B |g(x)| dx$$

for all balls B in $\mathbb{R}^n$. Show that there exists a constant c such that

$$\int_{\mathbb{R}^n} |f(x) - c| dx \leq 2 \int_{\mathbb{R}^n} |g(x)| dx.$$

(Of course, RHS may be infinite!)

Question 6 [10 marks]

Let $g : [0, \infty) \to [0, \infty)$ be a continuous non-decreasing concave function with $g(0) = 0$ and let $f_n \to f$ a.e. on a measurable set E. If $\int_E g(|f_n|) dx \leq M < \infty$ for all $n \in \mathbb{N}$. Show that for any measurable set $F \subset E$,

$$\lim_{n \to \infty} \int_F g(|f_n - f|) - g(|f_n|) + g(|f|) dx = 0.$$

Question 7 [10 marks]

Compute $\displaystyle\int_{-\infty}^{\infty} \frac{\sin 2x}{x} dx$.

Question 8 [7 marks]

Let $f : E \subset \mathbb{R} \to \mathbb{R}$ be a measurable function. If $g : [a, b] \to E$ is 1-1 with g^{-1} being absolutely continuous, show that $f \circ g : [a, b] \to \mathbb{R}$ is also measurable.

Question 9 [25 marks]

For at most **five** (5) of the following statements, prove or disprove each of the statement considered.

(a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that $\liminf_{x \to x_0} f(x) \geq f(x_0)$ for all $x_0 \in \mathbb{R}^n$, then f is Borel measurable.

(b) Let μ, ν be Borel measures on $\mathbb{R}^n$ such that $\nu(E) = 0$ whenever $\mu(E) = 0$ $E \subset \mathbb{R}^n$. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\nu(E) < \varepsilon$ whenever $\mu(E) < \delta$.

(c) Let $\{a_k\}$ be a sequence in $\mathbb{R}$ such that $\sum_{j=1}^{\infty} a_{\sigma(j)}$ converges for all permutations σ on $\mathbb{N}$. Then the series converges absolutely.

(d) Let a_n, r_n be two nonnegative sequences that converges to $a > 0$ and $r > 0$ respectively. Then $\lim_{n \to \infty} a_n^{r_n} = a^r$.

(e) Let f, g be nonnegative measurable functions on a measurable set E such that g is integrable on E. If

$$t \int_{\{f > t\}} g dx \leq C \quad \text{for all } t > 0$$

then

$$\int_E f^p g dx < \infty \quad \text{for all } 0 < p < 1.$$

(f) Consider a boundary value problem of a 2nd order linear ODE

$$y'' + p(x)y' + q(x)y = f(x), \quad y(a) = y_1, y(b) = y_2,$$

where $y_1, y_2 \in \mathbb{R}$. If all functions p, q, f are continuous on $[a, b]$, then exactly one of the following holds:

1. no solution;
2. unique solution;
3. infinite number of solutions.

(g) Let f be a continuous function on an open interval (a, b) such that

$$\int_a^b f \phi' dx = 0 \quad \text{for all } \phi \in C_c^\infty(a, b)$$

the space of infinitely differentiable functions with compact support in (a, b). Then f is a constant.