NATIONAL UNIVERSITY OF SINGAPORE

Ph.D. Qualifying Examination

Year 2021-2022 Semester I

Part I: Scientific Computing

Question 1 [20 marks]

Let $A, B \in \mathbf{R}^{4\times 4}$. Assume that $B^T A$ can be decomposed into

$$B^T A = U\Sigma V^T,$$

where $U, V \in \mathbf{R}^{4\times 4}$ are two known orthogonal matrices, and $\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Find an orthogonal matrix $Q \in \mathbf{R}^{4\times 4}$ such that

$$\|A - BQ\|_F \le \|A - BX\|_F$$

for all orthogonal matrices $X \in \mathbf{R}^{4\times 4}$.

Question 2 [15 marks]

Assume that the formula

$$I(f) = \sum_{i=0}^{6} A_i f(x_i)$$

approximating $\int_{-1}^{1} f(x)dx$ is exact for all polynomials of degree at most 6 and the distinct nodes x_i $(i = 0, 1, \cdots, 6)$ are symmetrically placed about the origin, compute the error

$$E(x^7) = \int_{-1}^{1} x^7 dx - \sum_{i=0}^{6} A_i x_i^7.$$

Question 3 [15 marks]

Consider solving numerically a well-posed initial value problem

$$\begin{aligned} &y' = f(t, y), \ a \le t \le b, \\ &y(a) = \alpha, \end{aligned}$$

using Taylor's method of order n, with step sizes h and $h/2$, respectively, where $h = \frac{b-a}{N}$ for some positive integer N. For $j = 0, 1, \cdots, N$, let y_j^h and $y_{2j}^{h/2}$ be the resulting approximations to $y(a + jh)$ using step sizes h and $h/2$, respectively. It is known that

$$y_j^h - y(a + jh) = \mathbf{O}(h^n), \ j = 0, 1, \cdots, N.$$

Show that the quantity

$$\frac{1}{2^n - 1}|y_j^h - y_{2j}^{h/2}|$$

gives an estimate for the absolute error in $y_{2j}^{h/2}$ for $j = 1, \cdots, N$.

Question 4 [15 marks]

Consider the differential equation

$$u_y + au_x = 0,$$

where a is a constant. Prove that $u(ih_x, jh_y)$ obtained by the characteristic method is the solution of the following finite difference scheme

$$u_{k,j+1} - u_{k,j} + \frac{as}{2}(u_{k+1,j} - u_{k-1,j}) - \frac{a^2 s^2}{2}(u_{k-1,j} - 2u_{k,j} + u_{k+1,j}) = 0, \ s = h_y/h_x,$$

when $h_y/h_x = 1/a$, where $h_x > 0$ and $h_y > 0$ are constants.

Part II: Optimization

1. [15 marks] Let $A \in R^{m\times n}$, $b \in R^m$ and $c \in R^n$.

   (a) Show that exactly one of the following two systems has a solution:

   S1: $Ax = b$, $x \geq 0$, $A^T y \leq c$, $c^T x - b^T y + \alpha = 0$ and $\alpha \geq 0$, for some $(x, y, \alpha) \in R^n \times R^m \times R$.

   S2: $A^T u + \gamma c \leq 0$, $Av - \gamma b = 0$, $b^T u + c^T v > 0$, $v \leq 0$ and $\gamma \leq 0$, for some $(u, v, \gamma) \in R^m \times R^n \times R$.

   (b) Consider the following primal and dual problems:

   $$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax = b, \ x \geq 0 \end{aligned}$$

   and

   $$\begin{aligned} \max \quad & b^T y \\ s.t. \quad & A^T y \leq c. \end{aligned}$$

   Show that both of the primal and dual problems have optimal solutions if and only if both of them are feasible.

2. [10 marks] Consider the following problem P:

   $$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & Ax \leq b, \end{aligned}$$

   where $f : R^n \to R$ is differentiable, $A \in R^{m\times n}$ and $b \in R^m$. Denote $A = \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$ where $A_i \in R^{1\times n}$. Let $\bar{x}$ be a feasible solution to Problem P. Denote $I = \{i : A_i \bar{x} = b_i\}$. Denote by A_I and b_I the submatrices consisting of I rows of A and b. Suppose the rows of A_I are linearly independent. Let $g = -\nabla f(\bar{x})$ and consider the following problem $\bar{P}$:

   $$\begin{aligned} \min \quad & \frac{1}{2}\|x - (\bar{x} + g)\|^2 \\ s.t. \quad & A_I x = b_I. \end{aligned}$$

(a) Write the KKT conditions for Problem $\bar{P}$.

(b) Determine a closed-form expression for the optimal solution $\hat{x}$ to Problem $\bar{P}$.

(c) Suppose that the given point $\bar{x}$ happens to be a KKT point for $\bar{P}$. Is $\bar{x}$ also a KKT point for P? If so, why? If not, under what additional conditions can you make this claim?

3. [10 marks] Consider the following problem:

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax = b \\ & x \in X \end{aligned}$$

where A is an $m \times n$-matrix and X consists of a finite number of points in R^n. For any $\lambda \in R^m$, define

$$\theta(\lambda) = \min\{c^T x + \lambda^T (Ax - b) : x \in X\}.$$

Assume that the set $\{x \in conv(X) : Ax = b\}$ is nonempty, where $conv(X)$ denotes the convex hull of X. Show that

$$\max\{\theta(\lambda) : \lambda \in R^m\} \quad = \quad \min\{c^T x : Ax = b,\ x \in conv(X)\}.$$

END OF PAPER