NATIONAL UNIVERSITY OF SINGAPORE, DEPARTMENT OF MATHEMATICS

Ph.D. Qualifying Examination
Year 2022-2023 Semester I

Time allowed : 3 hours

Part I: Scientific Computing

1. [15 marks]

   Suppose $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ is strictly row diagonally dominant. Prove that the Jacobi's method and the Gauss-Seidel method for linear system $Ax = b$ converges.

2. [15 marks]

   Give an explanation why the initial value problem

$$\frac{dy}{dx} = -\sqrt{|1-y^2|}, \qquad y(0) = 1,$$

   is not well-posed.

3. [20 marks]

   Let $h > 0$ and $x_n = nh$. Find out the order of the following numerical method

$$y_0 = 0, \quad y_{n+1} = (x_n + y_n + 1)e^h - (x_n + h + 1),$$

   applied to the differential equation

$$y' = y + x, \quad 0 \le x \le 1, \quad y(0) = 0.$$

4. [15 marks]

   The unconditionally stable finite difference scheme

$$\frac{h^2}{2\tau}(u_{i,j+1} - u_{i,j-1}) - (u_{i+1,j} - 2(\theta u_{i,j+1} + (1-\theta)u_{i,j-1}) + u_{i-1,j}) = 0 \tag{1}$$

   is used to approximate the equation

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0 \tag{2}$$

   at the grid point (x_i, t_j) with

$$x_i = ih, \quad t_j = j\tau, \quad \tau = \frac{1}{3}h^r, \quad r > 0,$$

   where $h > 0$ is the step size in x, r and θ are constants. Find out all θ and r such that the difference scheme (1) is consistent with the equation (2).

Part II: Optimization

1. [10 marks] Let A be a $p \times n$ matrix and B be a $q \times n$ matrix. Show that exactly one of the following systems has a solution:

   System 1: $Ax < 0$, $Bx = 0$ for some $x \in R^n$.

   System 2: $A^T u + B^T v = 0$ for some (u, v) with $u \neq 0$, $u \geq 0$.

2. [15 marks]

   (a) Let $f(x) = h(g(x)) = h(g_1(x), \ldots, g_k(x))$, where $h : R^k \to R$ is convex and nonincreasing, i.e. $h(u) \geq h(v)$ whenever $u \leq v$, and $g_i : R^n \to R$, $i = 1, \ldots, k$, are convex. (Here, no differentiability of h and g is assumed). Let $\mathrm{dom} f = \{x \in \mathrm{dom} g \mid g(x) \in \mathrm{dom} h\}$. Show that f is convex on its domain.

   (b) Show that $f(x) = -\sqrt{x_1 x_2}$ is convex in $x = (x_1, x_2) \in R^2_{++}$, where $R^k_{++} := \{x \in R^k \mid x > 0\}$.

   (c) Show that $f(x, u) = x^T x / u$ is convex in $(x, u) \in R^n \times R_{++}$.

   (d) Show that $f(x, u, v) = -\sqrt{uv - x^T x}$ is convex on $\mathrm{dom} f = \{(x, u, v) \mid uv > x^T x;\ x \in R^n;\ u, v \in R_{++}\}$.

3. [10 marks] Consider the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b, \end{array} \tag{3}$$

   where $f : R^n \to R$ is differentiable and convex, and $A \in R^{m \times n}$ with $\mathrm{rank} A = m$. Let

$$\phi(x) = f(x) + \alpha \|Ax - b\|_2^2,$$

   where $\alpha > 0$ is a parameter. Suppose $\bar{x}$ minimizes ϕ. Show that $f(\bar{x}) + 2\alpha \|A\bar{x} - b\|_2^2$ is a lower bound on the optimal value of the problem (3), that is,

$$f(\bar{x}) + 2\alpha \|A\bar{x} - b\|_2^2 \leq f(x)$$

   for all x satisfying $Ax = b$.

   (*Hint:* Find, from $\bar{x}$, a suitable feasible solution to the dual of (3).)